Internal Constructions in Homotopical Type Theory

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Abstract

The aim of this thesis is to investigate certain constructions that resist satisfactory full internalizations in plain homotopy type theory, i.e. intensional Martin-Löf type theory with the univalence axiom.

In Part I, we study internal higher categorical models of homotopical type theory, via wild categories with families (cwfs). We formulate coherence conditions on wild cwfs that suffice to recover properties expected of models of dependent type theory. The result is a definition of a 2-coherent wild cwf, which admits as instances both the syntax and the "standard model" given by a universe type. We also identify a higher "splitness" coherence condition that is satisfied by all set-level cwfs and univalent 2-coherent wild cwfs.

In Part II, we apply some of the theory developed in Part I and report on a partial investigation into the construction of type-valued Reedy-fibrant inverse diagrams in plain HoTT.

For

❧ *Anna,*

❧ *Mum,*

❧ *Dan, Sam and Ben,*

❧ *Nicolai,*

❧ *Tom, Brandon,*

❧ *my friends at the FP Lab,*

who have supported and encouraged me—Thank you.

Contents

Chapter 1

Prelude

1.1. ON FOUNDATIONS

1.1∗1. *Homotopical type theory.* This thesis is about *homotopical type theory*, a genre of Martin-Löf type theories with intensional identity types that do *not* assume the axiom of uniqueness of identity proofs (UIP). Theories in this genre may, but are not required to, admit anti-UIP principles such as univalence, and thus they include both intensional Martin-Löf type theory without UIP, as well as plain "book" homotopy type theory [Uni13].

1.1∗2. The majority of the results in this thesis are theorems and constructions in plain homotopy type theory (HoTT), occasionally with reasonable extensions such as inductive type families and small induction-recursion.

1.1∗3. However, as one underlying aim of this thesis is the investigation of categorical structures which are mathematically cogent but whose formal definition within plain HoTT is still conjectural, we do not intend for the reader of this thesis to treat HoTT, or indeed intensional type theory, as their sole metatheoretic "foundation". Instead, one should hold both the "traditional" (extensional, set-level aka"strict", proof-irrelevant) mathematical foundations alongside homotopical type theory as two complementary—and fruitfully interacting!—viewpoints toward the same underlying mathematical reality.

1.1∗4. While most of this thesis is intended to be readable in a "neutral" foundational setting, whether set theoretic or type theoretic, some parts only have novel content in an explicitly homotopical setting (e.g. Chapters 2 and 3 on wild categorical structures), while others must be read in settings with strict equality (e.g. Section 4.4). When relevant, we will explicitly indicate the background metatheory.

1.2. CONVENTIONS AND NOTATION

1.2∗1. *Universe types.* When we work in type theory we will assume "enough" universe types $\mathcal{U}$, as needed. Everything that we do in this thesis is independent of the exact details of the universe hierarchy; certainly the usual assumptions of successors and joins of universes suffice, as in e.g. [Rij22a, Sections 6.1 and 6.2]. As is common practice, we treat each $\mathcal{U}$ as though it were a Russell-style universe, while implicitly coercing to Tarski-style universes.

We also follow the convention of the HoTT book and use typally ambiguous universe notation. When we need to be explicit, we denote the successor of a universe $\mathcal{U}$ by $\mathcal{U}^+$.

1.2∗2. *Subuniverse of propositions.* The type of propositions in a universe $\mathcal{U}$ is denoted $\mathrm{Prop}_{\mathcal{U}}$, or simply Prop.

1.2∗3. *Functions.* We work with Π-types satisfying η and the type theoretic axiom of function extensionality.

1.2∗4. *Transport.* We denote the transport of an element $a : P(x)$ along an equality $e : x = y$ by

$$a{\downarrow}_e^P$$

(or $a{\downarrow}_e$ for conciseness), so that the expression

$$a = b{\downarrow}_e$$

may be read "a is equal to b over e", consistently with the previous notation.

1.2∗5. Additionally, the equation

$$a{\downarrow}_e^P{\downarrow}_{e'}^P = a{\downarrow}_{e \cdot e'}^P$$

holds by [Uni13, Lemma 2.3.9].

Part I

Internal Models of Homotopical Type Theory

Chapter 2

Wild Category Theory

2.1. WILD CATEGORIES

2.1∗1. Definition. *Wild categories.* In homotopical type theory, a *wild category* is like a precategory [AKS15, Definition 3.1] whose morphisms form *types* instead of sets. Explicitly, a wild category $\mathscr{C}$ consists of:

- A type $\mathscr{C}_0$ of objects.
- For all $x, y : \mathscr{C}_0$, a *type* $\mathscr{C}(x, y)$ of morphisms from x to y.
- An composition operation $\diamond$ of compatible morphisms, together with an associator α witnessing associativity of composition

  $$(h \diamond g) \diamond f \overset{\alpha_{f,g,h}}{=\!=\!=} h \diamond g \diamond f$$

  for all morphisms f, g, h.
- Identity morphisms id_x for all $x : \mathscr{C}_0$, together with unitors λ and ρ witnessing the left and right identity equations

  $$\mathrm{id}_y \diamond f \overset{\lambda_f}{=\!=} f$$
  $$f \diamond \mathrm{id}_x \overset{\rho_f}{=\!=} f$$

  for all $f : \mathscr{C}(x, y)$.

Crucially, no further coherence laws on hom-types are required in the base definition of a wild category.

2.1∗2. We denote identity morphisms id_x by id, and the coherators $\alpha_{f,g,h}$, λ_f and ρ_f by α, λ and ρ when the relevant arguments are inferrable from context.

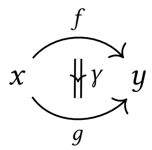
2.1∗3. Examples. *Wild categories.* We are primarily interested in two particular classes of wild category, which are in fact completely coherent. These are distinguished by the behavior of their hom-types: the "maximally truncated" and the "nontrivially fully coherent".

In the first class are the wild categories whose morphisms form sets: any precategory (and hence, set-level or univalent 1-category) is immediately a wild category.

The second class consists of the type theoretic universes. If $\mathscr{U}$ is a universe of a type theory with Π-types, there is a wild category $\mathcal{U}$ with objects $\mathcal{U}_0 :\equiv \mathscr{U}$, and whose hom-types are the function types $\mathcal{U}(A, B) :\equiv A \longrightarrow B$. Composition is given by function composition and identity morphisms by identity functions. The associativity and unit laws hold definitionally, i.e. α, λ and ρ are families of trivial equalities.

2.1∗4. Example. *Wild categories from subuniverses.* The definition of a universe wild category (2.1∗3) applies equally well to any reflective subuniverse $\Sigma\, \mathscr{U}\, P$ [Uni13, Definition 7.7.1],[1] In this way, the m-modal types for any modality m on a universe $\mathscr{U}$ [Uni13, Section 7.7; RSS20, Section 1] form a wild category. In particular, we have wild categories whose objects are the n-types in $\mathscr{U}$, and which might be seen as prototypical examples of wild $(n + 1, 1)$-categories.

2.1∗5. We draw diagrams in wild categories after the familiar 1-categorical notation, with one important difference. Since morphisms in a wild category can form arbitrary types, equality of morphisms—and in particular, commutativity of a given face of a diagram—is no longer property, but *data*. Hence we denote commuting faces with the notation



to make explicit the commutativity witness $\gamma : f = g$.

2.1∗6. As the notation (2.1∗5) suggests, we call equalities between morphisms in a wild category "2-cells" and higher equalities "higher cells", in the expected manner. We stress, however, that these do not come from any explicitly axiomatized categorical structure, but instead arise out of the ambient (homotopical) type theory.

2.1∗7. We are led to distinguish the *categorical* and *typal* directions of a wild category: the 1-cells belong to the categorical direction, and the higher cells to the typal. This recalls the point of view—implicit in Rezk [Rez01] and explicated by Joyal and Tierney [JT07]—that Segal spaces (i.e. bisimplicial sets $\Delta^{op} \times \Delta^{op} \longrightarrow \mathrm{Set}$ satisfying the Segal condition) have categorical and *spatial* directions.

[1] Which should straightforwardly yield a notion of "wild reflective subcategory"; we do not develop this.

2.1∗8. *On the semantics of wild categories.* We have defined wild categories by presenting them using a generalized algebraic theory in homotopical type theory. The fact that a wild category is a strict generalization of a precategory thus relies crucially on the homotopical nature of the identity type in type theory. It is therefore natural to ask if wild categories can be given semantics in other, perhaps more traditional, mathematical settings.

From the observation (2.1∗6) one might guess that wild categories can be modeled by Segal spaces "restricted to 1-categorical structure in the categorical direction", i.e. by functors $\Delta^{\mathrm{op}}_{\leq n} \times \Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}$ for some small n, satisfying an appropriately restricted Segal condition. Indeed, Capriotti and Kraus [CK17, Theorem 4.9] show (a version of) this to hold.[2]

Alternatively, following van den Berg and Garner [BG11] in viewing types as Batanin-Leinster weak ω-groupoids, we conjecture that we may also consider wild categories to be categories "wildly enriched"[3] in weak ω-groupoids.

2.1∗9. One might question the point of wild categories since, satisfying no higher coherence laws in general, they can indeed be very wildly behaved (for instance, λ_{id_x} and ρ_{id_x} may be distinct witnesses of $\mathrm{id}_x \diamond \mathrm{id}_x = \mathrm{id}_x$ without assuming a triangle coherator). This impedes many further constructions one would typically want to perform; as a standard example, the usual construction of the slice category $\mathscr{C}/c$ does not have associative composition unless $\mathscr{C}$ has pentagon coherators.

2.1∗10. In reply to the potential objections raised by (2.1∗9), we offer two observations:

1. Wild categories are a common generalization of pre-, higher, and ∞-categories. In particular, many structures that are *externally* recognized to have the structure of an ∞-category are also examples of wild categories in homotopical type theory.[4] Crucially, the type of wild categories is definable in homotopical type theory, while a uniform definition of the type of $(n, 1)$-categories and ∞-categories remains elusive.
2. Many of the further categorical constructions we want to perform can be controlled and studied by assuming only a *finite* number of nontrivial coherators. By taking this à la carte approach we can develop a useful amount of theory in "sufficiently coherent" wild categories, which we may then use as a stepping stone to further investigate fully coherent higher structures in type theory.

We illustrate these two points by developing a theory of pullbacks in wild categories (Section 2.6), which we later use to organize the construction of inverse diagrams in wild internal models of homotopical type theory. In particular, it turns out

[2] By interpreting their type theoretic proof in the simplicial model of HoTT [KL21].

[3] That is, weakly enriched, but without requiring any coherences on n-cells for $n \geq 2$.

[4] Morally, there is a forgetful functor from ∞-categories to wild categories, which we expect to be able to define in a setting such as two-level type theory (2LTT). This would imply that *all* fibrant structures (in the 2LTT sense) that are recognized as ∞-categories on the outer level are recognized as wild categories in the inner level.

that the existence of coherators for 2-cells suffice for many of our purposes. Thus, the theory developed in Part I is directly applicable to categorical structures with sufficiently coherent hom-1-types[5]. This last observation also suggests the following question.

2.1∗11. Question. A third class of wild category, sitting in between the two classes of examples in (2.1∗3), are the wild categories with hom-n-types, which, intuitively, should correspond to "precoherent" $(n, 1)$-categories. Could the theory of wild categories, together with the theory of n-types, give a useful perspective from which to study coherent $(n, 1)$-categories? We leave this question unexplored in this thesis.

2.2. BASIC CONCEPTS

Expectedly, a large number of elementary concepts from univalent 1-category theory [AKS15] and bicategory theory [Ahr+21] straightforwardly generalize to (and are subsumed by) the wild categorical setting.

2.2∗1. *Terminal objects.* An object y in a wild category $\mathscr{C}$ is *terminal* if $\mathscr{C}(x, y)$ is contractible for all objects $x : \mathscr{C}_0$.

2.2∗2. *Sections and retractions.* The type of *sections* of a morphism $f : \mathscr{C}(x, y)$ in a wild category $\mathscr{C}$ is

$$\mathrm{Sect}(f) :\equiv \textstyle\sum_{(s\,:\,\mathscr{C}(y,\,x))},\ f \diamond s = \mathrm{id}.$$

Similarly, the type of *retractions* of f is

$$\mathrm{Retr}(f) :\equiv \textstyle\sum_{(r\,:\,\mathscr{C}(y,\,x))},\ r \diamond f = \mathrm{id}.$$

In contrast to 1-categories, sections and retractions in wild categories are not solely determined by their morphism component, but also by the identification of the section-retraction composite with the identity.

2.2∗3. *Whiskering.* Given an equality $\gamma : g = g'$ of morphisms $g, g' : \mathscr{C}(x, y)$, for any morphism $f : \mathscr{C}(w, x)$ the *right whiskering* $(\gamma * f)$ of γ with f is the canonical equality

$$\mathrm{ap}\,(\_ \diamond f)\,\gamma : g \diamond f = g' \diamond f,$$

and for any $h : \mathscr{C}(y, z)$ the *left whiskering* $(h * \gamma)$ of γ with h is the equality

$$\mathrm{ap}\,(h \diamond \_)\,\gamma : h \diamond g = h \diamond g'.$$

2.2∗4. By induction, the following equations hold for right whiskering:

$$\begin{aligned} \mathrm{refl} * f &\equiv \mathrm{refl}, \\ \gamma * \mathrm{id} &= \rho \cdot \gamma \cdot \rho^{-1}, \\ (\gamma * f)^{-1} &= \gamma^{-1} * f, \\ (\gamma \cdot \delta) * f &= (\gamma * f) \cdot (\delta * f), \end{aligned}$$

[5] Viz., morally, (2, 1)-categories and (2, 1)-categories with families.

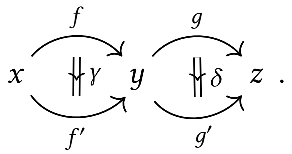
with the analogous equations for left whiskering. We also have the following associativity laws expressing “naturality” of α,

$$\begin{aligned} g * (f * \gamma) &= \alpha^{-1} \cdot ((g \diamond f) * \gamma) \cdot \alpha, \\ (\gamma * g) * f &= \alpha \cdot (\gamma * (g \diamond f)) \cdot \alpha^{-1}, \\ (g * \gamma) * f &= \alpha \cdot (g * (\gamma * f)) \cdot \alpha^{-1}, \end{aligned}$$

and the interchange law

$$(g * \gamma) \cdot (\delta * f') = (\delta * f) \cdot (g' * \gamma)$$

for all morphisms f, f', g, g' and equalities γ, δ as in



2.3. COHERENCES FOR 2-CELLS

2.3∗1. The following two coherence conditions are familiar from the definition of a bicategory.

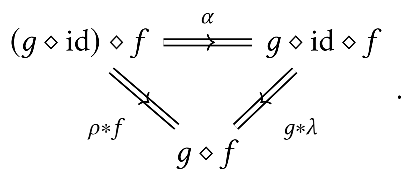
2.3∗2. Definition. *Triangle coherators.* A wild category $\mathscr{C}$ has *triangle coherators for unitors* if for all morphisms

$$x \xrightarrow{f} y \xrightarrow{g} z$$

in $\mathscr{C}$, there is an equality

$$\triangle_{f,g} : \alpha \cdot (g * \lambda) = \rho * f$$

filling the triangle



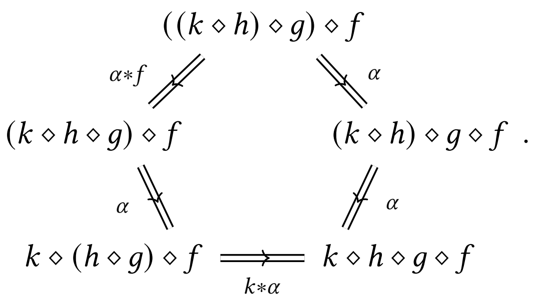
2.3∗3. Definition. *Pentagonators.* $\mathscr{C}$ has *pentagon coherators for associators,* or *(α-) pentagonators,* if for all sequences

$$v \xrightarrow{f} w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{k} z$$

of morphisms in $\mathscr{C}$, there is an equality

$$⬠_{f,g,h,k} : (\alpha * f) \cdot \alpha \cdot (k * \alpha) = \alpha \cdot \alpha$$

filling the usual pentagon



2.3∗4. Definition. *2-coherent wild categories.* A wild category is called *2-coherent* if it has triangle and pentagon coherators.

2.3∗5. Examples. Any precategory trivially satisfies all higher equalities between equalities of morphisms. The universe wild categories $\mathcal{U}$ have definitionally associative composition of morphisms, and thus have trivial triangle and pentagon coherators. Thus precategories and universes are 2-coherent wild categories.

2.3∗6. *2-coherent wild categories in the literature.* Hart and Hou [HH24] call 2-coherent wild categories *bicategories*; we have chosen not to use this terminology as 2-coherent wild categories always have invertible 2-cells and are thus closer in spirit to (2, 1)-categories.

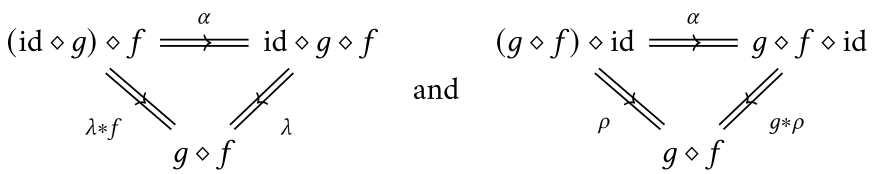
2-coherent wild categories are also essentially the *wild 2-precategories* of Capriotti and Kraus [CK17]; the difference is that they also include the other two triangle coherators of (2.3∗8) in the data of the type.

We also have the following link to the univalent bicategory theory of Ahrens et al. [Ahr+21]:

2.3∗7. Any 2-coherent wild category is also a *prebicategory* in the sense of Ahrens et al. [Ahr+21, Definition 2.1], by taking the type of 2-cells from f to g to be the equality type $f = g$, and using (2.2∗4).

2.3∗8. Many coherences involving λ, ρ and α that hold in all bicategories also hold in 2-coherent wild categories—namely, those that do not rely on uniqueness of equality of 2-cells. In particular, in a 2-coherent wild category $\mathscr{C}$, there are witnesses (not necessarily unique) that

- $\lambda_{\mathrm{id}_x} = \rho_{\mathrm{id}_x}$ for all $x : \mathscr{C}_0$, and
- the diagrams of equalities



commute for all $f : \mathscr{C}(x, y)$ and $g : \mathscr{C}(y, z)$.

We refer to [JY21, Propositions 2.2.4 and 2.2.6] for proofs of these facts.

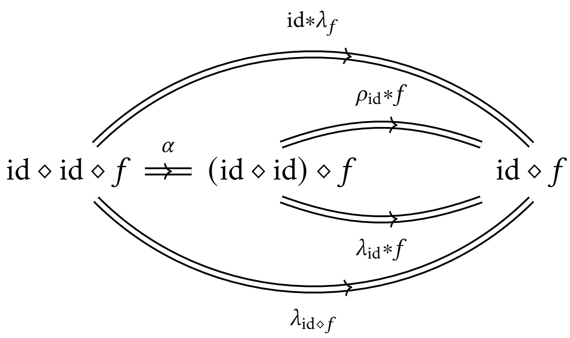
2.3∗9. Non-example. *A wild category missing coherators.* In [Kra21, Lemma 8], Kraus uses the circle higher inductive type to construct a wild category $\mathscr{E}$ with an object $x : \mathscr{E}_0$ that refutes $\lambda_{\mathrm{id}_x} = \rho_{\mathrm{id}_x}$. By (2.3∗8), $\mathscr{E}$ must therefore also fail to have triangle or pentagon coherators. In fact, one can see directly by [Uni13, Lemma 6.4.2] that the existence of triangle coherators for $\mathscr{E}$ is a HoTT taboo.

2.3∗10. Lemma. Suppose that $\gamma : g = \mathrm{id} \diamond f$ in a 2-coherent wild category. Then

$$\begin{array}{ccc} \mathrm{id} \diamond g & \overset{\mathrm{id}*\gamma}{=\!=\!=} & \mathrm{id} \diamond \mathrm{id} \diamond f \\ {\scriptstyle\lambda}\Vert & & \Vert{\scriptstyle \mathrm{id}*\lambda} \\ g & \underset{\gamma}{=\!=\!=} & \mathrm{id} \diamond f \end{array}$$

commutes, i.e. $\lambda_g \cdot \gamma =_{(\mathrm{id}\,\diamond\, g\, =\, \mathrm{id}\,\diamond\, f)} (\mathrm{id} * \gamma) \cdot (\mathrm{id} * \lambda_f)$.

Proof. We equivalently prove that $\mathrm{id}*\gamma = \lambda_g \cdot \gamma \cdot (\mathrm{id} * \lambda_f)^{-1}$. Since $\mathrm{id}*\gamma = \lambda_g \cdot \gamma \cdot \lambda^{-1}_{\mathrm{id}\diamond f}$ by properties of whiskering (2.2∗4), the result follows if $\mathrm{id} * \lambda_f = \lambda_{\mathrm{id}\diamond f}$. The following pasting diagram shows how to construct such an equality: the interior of



is divided into two triangles and a bigon, which commute by the triangle coherator and the equalities in (2.3∗8). □

2.4. EQUIVALENCE, ISOMORPHISM AND UNIVALENCE

2.4∗1. Definition. *Wild equivalences.* A morphism $f : \mathscr{C}(x, y)$ in a wild category $\mathscr{C}$ is a *wild equivalence* if it has both a section and a retraction (i.e. is biinvertible).

We denote the type of wild equivalences from x to y in $\mathscr{C}$ by $x \simeq_{\mathscr{C}} y$. To avoid confusion with type theoretic equivalences, we also call elements of $x \simeq_{\mathscr{C}} y$ *$\mathscr{C}$-equivalences*.

2.4∗2. *The universal property of wild equivalences.* A morphism $f : \mathscr{C}(x, y)$ in a wild category $\mathscr{C}$ is a $\mathscr{C}$-equivalence if and only if it is *neutral*—i.e. if and only if for any $w, z : \mathscr{C}_0$, the maps

$$\begin{aligned} f \diamond \_ &: \mathscr{C}(w, x) \to \mathscr{C}(w, y) \\ \text{and} \quad \_ \diamond f &: \mathscr{C}(y, z) \to \mathscr{C}(x, z) \end{aligned}$$

are equivalences of hom-types.

Proof. Let s and r be, respectively, a section and retraction of f in $\mathscr{C}$. Then $(s \diamond \_)$ and $(r \diamond \_)$ are, respectively, sections and retractions of $(f \diamond \_)$, while $(\_ \diamond r)$ and $(\_ \diamond s)$ are, respectively, sections and retractions of $(\_ \diamond f)$.

Conversely, the inverse equivalences of $f \diamond \_ : \mathscr{C}(y, x) \xrightarrow{\sim} \mathscr{C}(y, y)$ and $\_ \diamond f : \mathscr{C}(y, x) \xrightarrow{\sim} \mathscr{C}(x, x)$ map identity morphisms to a section and retraction of f, respectively. □

2.4∗3. Capriotti and Kraus [CK17, Lemma 5.3] observe, as an immediate corollary of (2.4∗2), that being a wild equivalence is a proposition and hence that (2.4∗2) improves to an equivalence. Note that the definition of neutrality of morphisms is sensitive to the universe level of the type of objects, while that of biinvertibility is not. This may be relevant when considering e.g. locally small categories.

2.4∗4. *Wild isomorphisms.* We could also consider the type of wild *isomorphisms* $x \cong_{\mathscr{C}} y$ in $\mathscr{C}$, defined to be the type of morphisms $f : \mathscr{C}(x, y)$ having a two-sided inverse $g : \mathscr{C}(y, x)$.

2.4∗5. *Wild isomorphism implies wild equivalence.* Any two-sided inverse is both a section and a retraction, so for any objects $x, y : \mathscr{C}_0$ there is a canonical map

$$\mathrm{isotoeqv}_{\mathscr{C}} : (x \cong_{\mathscr{C}} y) \longrightarrow (x \simeq_{\mathscr{C}} y).$$

If $\mathscr{C}$ is a precategory then $\mathrm{isotoeqv}_{\mathscr{C}}$ is an equivalence—it has an inverse that sends a section-retraction pair (s, r) of f to the two-sided inverse $r \diamond f \diamond s$ of f.

2.4∗6. Definition. *Wild isomorphisms from equality.* If $x, y : \mathscr{C}_0$ are objects of a wild category such that $e : x = y$, there is a morphism[6]

$$\mathrm{idd}(e) :\equiv \mathrm{id}_x \downarrow_e^{\mathscr{C}(x,\_)} : \mathscr{C}(x, y).$$

By induction on e, $\mathrm{idd}(e)$ is an isomorphism, with inverse $\mathrm{idd}(e^{-1})$.

2.4∗7. Definition. *Equality implies wild equivalence.* idd thus gives us a map

$$x = y \longrightarrow x \cong_{\mathscr{C}} y.$$

We postcompose this with $\mathrm{isotoeqv}_{\mathscr{C}}$ (2.4∗5) to define

$$\mathrm{idtoeqv}_{\mathscr{C}} : x = y \longrightarrow x \cong_{\mathscr{C}} y \longrightarrow x \simeq_{\mathscr{C}} y.$$

2.4∗8. Definition. *Univalent wild categories.* A *univalent wild category* is a wild category $\mathscr{C}$ for which $\mathrm{idtoeqv}_{\mathscr{C}}$ is an equivalence. That is, "wild equivalence is equality" of objects in univalent wild categories.

[6] idd for "*id*entity morphism *d*ependent over an equality".

2.4∗9. Thus if $\mathscr{C}$ is univalent, we get an inverse

$$\mathrm{ua}_{\mathscr{C}} : x \simeq_{\mathscr{C}} y \to x = y$$

of $\mathrm{idtoeqv}_{\mathscr{C}}$. In particular,

$$\mathrm{idtoeqv}_{\mathscr{C}}(\mathrm{ua}_{\mathscr{C}}(f, u)) = (f, u)$$

for all morphisms $f : \mathscr{C}(x, y)$ with proofs u witnessing that f is a $\mathscr{C}$-equivalence.

2.4∗10. Lemma. idd *"retracts"* $\mathrm{ua}_{\mathscr{C}}$. Projecting out the first components of the equality in (2.4∗9), we have that

$$\mathrm{idd}(\mathrm{ua}_{\mathscr{C}}(f, u)) = f$$

for any $\mathscr{C}$-equivalence (f, u).

2.4∗11. *Examples and non-examples.* Univalent 1-categories and universe wild categories are univalent wild categories. An arbitrary precategory is not generally a univalent wild category; in particular, any set-level univalent category must be gaunt.

2.5. COMMUTING SQUARES

We can formulate the notion of a limit in a wild category $\mathscr{C}$ by analogy with the 1-categorical case, as terminal cones over diagrams in $\mathscr{C}$. However, for the applications in Part II of this thesis it is convenient to define commuting squares and pullbacks in wild categories directly.

2.5∗1. Definition. *Cospans.* A *cospan* in a wild category $\mathscr{C}$ is an element of

$$\mathrm{Cospan}(\mathscr{C}) :\equiv \sum_{(A, B, C \,:\, \mathscr{C}_0)} (f : \mathscr{C}(A, C))\, (g : \mathscr{C}(B, C)),$$

consisting of three objects $A, B, C : \mathscr{C}_0$ and two arrows $f : \mathscr{C}(A, C)$, $g : \mathscr{C}(B, C)$. We denote cospans (A, B, C, f, g) by $A \xrightarrow{f} C \xleftarrow{g} B$, or even simply (f, g) when the vertices are understood.

2.5∗2. Definition. *Commuting squares.* A *commuting square* in a wild category $\mathscr{C}$ consists of a cospan $A \xrightarrow{f} C \xleftarrow{g} B$, an object $X : \mathscr{C}_0$, two morphisms $m_A : \mathscr{C}(X, A)$ and $m_B : \mathscr{C}(X, B)$, and an equality $\gamma : f \diamond m_A = g \diamond m_B$, as in the following diagram in $\mathscr{C}$,

$$\begin{array}{ccc} X & \xrightarrow{m_B} & B \\ {\scriptstyle m_A}\downarrow & \not\!\!/\!/_{\gamma} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}.$$

2.5∗3. Definition. *Commuting squares with fixed source.* The type of commuting squares in $\mathscr{C}$ is indexed over cospans and source objects. That is, for each cospan $\mathfrak{c} :\equiv A \xrightarrow{f} C \xleftarrow{g} B$ and object $X : \mathscr{C}_0$ there is a type

$$\mathrm{CommSq}_{\mathfrak{c}}(X) :\equiv \sum_{(m_A \,:\, \mathscr{C}(X, A))\, (m_B \,:\, \mathscr{C}(X, B))} (\gamma : f \diamond m_A = g \diamond m_B)$$

of *commuting squares on* $\mathfrak{c}$ *with source* X.

2.5∗4. Observe that $\mathrm{CommSq}_{\mathfrak{c}}(X)$ is the type theoretic pullback [AKL15, Definition 4.11] given by

$$\begin{array}{ccc} \mathrm{CommSq}_{\mathfrak{c}}(X) & \longrightarrow & \mathscr{C}(X, B) \\ \downarrow & & \downarrow {\scriptstyle g \diamond \_} \\ \mathscr{C}(X, A) & \xrightarrow[f \diamond \_]{} & \mathscr{C}(X, C) \end{array} .$$

Avigad, Kapulkin and Lumsdaine [AKL15] also call $\mathrm{CommSq}_{\mathfrak{c}}(X)$ (instantiated in universes) the type of *cones* over $\mathfrak{c}$ with vertex X. We will not use this terminology since it suggests we also want a morphism $X \to C$.

2.5∗5. $\mathrm{CommSq}_{\mathfrak{c}}(X)$ *is a set in precategories.* If $\mathscr{C}$ is a precategory, the type $\mathrm{CommSq}_{\mathfrak{c}}(X)$ of (2.5∗3) is a Σ-type of sets and propositions, and thus also a set.

2.5∗6. To characterize the equality of commuting squares, we recall the following version of the *fundamental theorem of identity types* [7].

2.5∗7. Theorem. *Fundamental theorem of identity types.* Assume a pointed type (A, a), a type family R over A, and an element $r : R(a)$. The canonical family of maps

$$\prod_{(x\,:\,A)} (a = x) \to R(x)$$

is a family of equivalences if and only if the total space $\Sigma\, A\, R$ is contractible.

For a proof, we refer to [Rij22a, Theorem 11.2.2].

2.5∗8. Lemma. *Equality of* $\mathrm{CommSq}_{\mathfrak{c}}(X)$. Let $\mathfrak{c} \equiv A \xrightarrow{f} C \xleftarrow{g} B$ be a cospan, $X : \mathscr{C}_0$ an object in $\mathscr{C}$, and $\mathfrak{S} \equiv (m_A, m_B, \gamma)$ and $\mathfrak{S}' \equiv (m_A{}', m_B{}', \gamma')$ commuting squares on $\mathfrak{c}$ with source X, i.e. elements of $\mathrm{CommSq}_{\mathfrak{c}}(X)$.

The equality type $\mathfrak{S} = \mathfrak{S}'$ is equivalent to

$$\sum_{(e_A\,:\,m_A = m_A{}')\,(e_B\,:\,m_B = m_B{}')}, \gamma = (f * e_A) \cdot \gamma' \cdot (g * e_B)^{-1},$$

where the last component is equivalent to the type of equalities filling the square

$$\begin{array}{ccc} f \diamond m_A & \overset{\gamma}{=\!=} & g \diamond m_B \\ {\scriptstyle f * e_A} \Vert & & \Vert {\scriptstyle g * e_B} \\ f \diamond m_A{}' & \underset{\gamma'}{=\!=} & g \diamond m_B{}' \end{array} .$$

Proof. Apply the fundamental theorem of identity types (2.5∗7) to the pointed type $(\mathrm{CommSq}_{\mathfrak{c}}(X), \mathfrak{S})$ and the family R over $\mathrm{CommSq}_{\mathfrak{c}}(X)$ given by

$$R\,(k_A, k_B, \delta) :\equiv \sum_{(e_A\,:\,m_A = k_A)\,(e_B\,:\,m_B = k_B)}, \gamma = (f * e_A) \cdot \delta \cdot (g * e_B)^{-1}$$

[7] So named by Rijke, who recognized its utility in [Rij22a, Chapter 11].

and pointed at $(\mathrm{refl}_{m_A}, \mathrm{refl}_{m_B}, \mathrm{refl}_\gamma) : R(\mathfrak{S})$. It's thus enough to show that

$$\sum \mathrm{CommSq}_{\mathrm{c}}(X)\, R$$

is contractible. By the algebra of Σ-types this is equivalent to

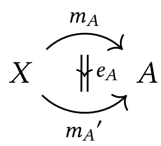
$$\begin{aligned}(k_A : \mathscr{C}(X, A)) \times (e_A : m_A = k_A)\\ \times (k_B : \mathscr{C}(X, B)) \times (e_B : m_B = k_B)\\ \times (\delta : f \diamond k_A = g \diamond k_B) \times (\gamma = (f * e_A) \cdot \delta \cdot (g * e_B)^{-1}),\end{aligned}$$

which is further equivalent to

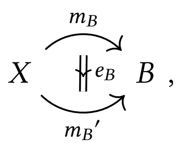
$$\sum (\delta : f \diamond m_A = g \diamond m_B),\ \gamma = \delta$$

by contracting singletons and (2.2∗4). This last type is also contractible. □

2.5∗9. By the previous characterization (2.5∗8), we can understand an equality of the commmuting squares $\mathfrak{S}$, $\mathfrak{S}' : \mathrm{CommSq}_{\mathrm{c}}(X)$ to consist of two 2-cell bigons



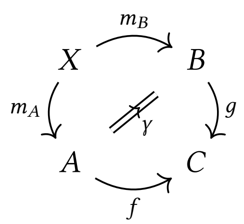
and



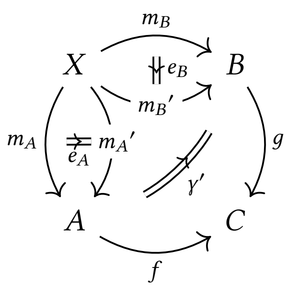
,

and a 3-cell volume filling the interior of the "sphere" whose top hemisphere is

$\mathfrak{S} \equiv$



and whose bottom hemisphere is the pasting



of e_A, e_B and $\mathfrak{S}'$.

2.5∗10. *Transpose of commuting squares.* For any $X : \mathscr{C}_0$ and cospan $A \xrightarrow{f} C \xleftarrow{g} B$, there is a map

$$\_^{\mathrm{T}} : \mathrm{CommSq}_{(f,g)}(X) \to \mathrm{CommSq}_{(g,f)}(X)$$

defined by

$$(m_A, m_B, \gamma)^{\mathrm{T}} :\equiv (m_B, m_A, \gamma^{-1}).$$

We call $\mathfrak{S}^{\mathrm{T}}$ the *transpose* of the commuting square $\mathfrak{S}$.

2.5∗11. *Transpose is involutive.* $(\mathfrak{S}^{\mathrm{T}})^{\mathrm{T}} = \mathfrak{S}$ for all $\mathfrak{S}$, and so in particular $\_^{\mathrm{T}}$ is an equivalence.

2.5∗12. *Canonical horizontal pasting.* Suppose

$$\begin{array}{ccccc} A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \\ {\scriptstyle i}\downarrow & \not\Rightarrow_{\mathfrak{q}} & {\scriptstyle j}\downarrow & \not\Rightarrow_{\mathfrak{p}} & \downarrow{\scriptstyle k} \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C \end{array}$$

is a diagram in $\mathscr{C}$ with commuting inner squares $\mathfrak{Q} :\equiv (i, f', \mathfrak{q})$ and $\mathfrak{P} :\equiv (j, g', \mathfrak{p})$ on the respective cospans. The *(canonical) horizontal pasting* $\mathfrak{Q} \mid \mathfrak{P}$ of $\mathfrak{Q}$ and $\mathfrak{P}$ is the commuting square

$$\mathfrak{Q} \mid \mathfrak{P} \;:\equiv\; \begin{array}{ccc} A' & \xrightarrow{g' \diamond f'} & C' \\ {\scriptstyle i}\downarrow & \not\Rightarrow_{\mathfrak{q}|\mathfrak{p}} & \downarrow{\scriptstyle k} \\ A & \xrightarrow[g \diamond f]{} & C \end{array}$$

where

$$\mathfrak{q} \mid \mathfrak{p} \;:\equiv\; \alpha \cdot (g * \mathfrak{q}) \cdot \alpha^{-1} \cdot (\mathfrak{p} * f') \cdot \alpha.$$

2.5∗13. *Canonical vertical pasting.* Similarly, if

$$\begin{array}{ccc} A' & \xrightarrow{i} & A \\ {\scriptstyle f'}\downarrow & \not\Rightarrow_{\mathfrak{q}} & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{j} & B \\ {\scriptstyle g'}\downarrow & \not\Rightarrow_{\mathfrak{p}} & \downarrow{\scriptstyle g} \\ C' & \xrightarrow[k]{} & C \end{array}$$

is a diagram in $\mathscr{C}$ with commuting inner squares $\mathfrak{Q} :\equiv (f', i, \mathfrak{q})$ and $\mathfrak{P} :\equiv (g', j, \mathfrak{p})$, the *(canonical) vertical pasting* $\frac{\mathfrak{Q}}{\mathfrak{P}}$ of $\mathfrak{Q}$ and $\mathfrak{P}$ is

$$\frac{\mathfrak{Q}}{\mathfrak{P}} \;:\equiv\; \begin{array}{ccc} A' & \xrightarrow{i} & A \\ {\scriptstyle g' \diamond f'}\downarrow & \not\Rightarrow_{\frac{\mathfrak{q}}{\mathfrak{p}}} & \downarrow{\scriptstyle g \diamond f} \\ C' & \xrightarrow[k]{} & C \end{array}$$

where

$$\frac{\mathfrak{q}}{\mathfrak{p}} \;:\equiv\; \alpha^{-1} \cdot (\mathfrak{p} * f') \cdot \alpha \cdot (g * \mathfrak{q}) \cdot \alpha^{-1}.$$

2.5∗14. Definition. *Vertical pasting map.* Restricting the diagram in (2.5∗13), suppose that

$$\mathfrak{P} \;:\equiv\; \begin{array}{ccc} B' & \xrightarrow{j} & B \\ {\scriptstyle g'}\downarrow & \not\Rightarrow_{\mathfrak{p}} & \downarrow{\scriptstyle g} \\ C' & \xrightarrow[k]{} & C \end{array} .$$

is a commuting square. For any $A : \mathscr{C}_0$, $f : \mathscr{C}(A, B)$ and $X : \mathscr{C}_0$, the canonical vertical pasting with $\mathfrak{P}$ (2.5∗13) yields a map $\mathrm{CommSq}_{(j,f)}(X) \to \mathrm{CommSq}_{(k,g\diamond f)}(X)$. That is, we have the family

$$\frac{\_}{\mathfrak{P}} : \prod (A : \mathscr{C}_0)\, (f : \mathscr{C}(A, B))\, (X : \mathscr{C}_0)\ \mathrm{CommSq}_{(j,f)}(X) \to \mathrm{CommSq}_{(k,g\diamond f)}(X).$$

2.5∗15. Definition (2.5∗13) is in some ways redundant: if $\mathfrak{P}$ and $\mathfrak{Q}$ are commuting squares as in (2.5∗13) then a straightforward calculation shows that

$$\frac{\mathfrak{Q}}{\mathfrak{P}} = (\mathfrak{Q}^{\mathsf{T}} \mid \mathfrak{P}^{\mathsf{T}})^{\mathsf{T}},$$

and so one could simply vertically paste squares by horizontally pasting their transposes. However, this equality is only typal, and it turns out to be more convenient for later proofs and constructions (and computer mechanizations) to use the particular form of the canonical vertical pasting as we have defined it.

2.5∗16. Definition. *Precomposing squares with morphisms.* Morphisms in $\mathscr{C}$ act contravariantly on commuting squares by precomposition at their source. Assume that $\mathfrak{c} \equiv A \xrightarrow{f} C \xleftarrow{g} B$ as before, and that $X, Y : \mathscr{C}_0$. There is a function

$$\_ \mathbin{\square} \_ : \mathrm{CommSq}_{\mathfrak{c}}(Y) \to \mathscr{C}(X, Y) \to \mathrm{CommSq}_{\mathfrak{c}}(X)$$

sending

$$\begin{array}{ccc} Y & \xrightarrow{m_B} & B \\ {\scriptstyle m_A}\downarrow & \gamma & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \quad\mapsto\quad X \xrightarrow{m} Y \quad\mapsto\quad \begin{array}{ccc} X & \xrightarrow{m_B\diamond m} & B \\ {\scriptstyle m_A\diamond m}\downarrow & \delta & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array},$$

where δ is the composition of paths

$$f \diamond m_A \diamond m \overset{\alpha^{-1}}{=\!=} (f \diamond m_A) \diamond m \overset{\gamma * m}{=\!=} (g \diamond m_B) \diamond m \overset{\alpha}{=} g \diamond m_B \diamond m$$

given by conjugating the right whiskering of γ with m by the associator α of $\mathscr{C}$. That is,

$$(m_A, m_B, \gamma) \mathbin{\square} m \;\;:\equiv\;\; (m_A \diamond m,\ m_B \diamond m,\ \alpha^{-1} \cdot (\gamma * m) \cdot \alpha).$$

2.5∗17. Lemma. *Right action of morphisms on commuting squares.* If $\mathfrak{S} : \mathrm{CommSq}_{\mathfrak{c}}(X)$ is a commuting square in a 2-coherent wild category (2.3∗4) then

$$\mathfrak{S} \mathbin{\square} \mathrm{id}_X = \mathfrak{S},$$

and for all $f : \mathscr{C}(X, Y)$ and $g : \mathscr{C}(Y, Z)$,

$$\mathfrak{S} \mathbin{\square} (g \diamond f) = \mathfrak{S} \mathbin{\square} g \mathbin{\square} f.$$

Proof. By calculation, properties of whiskering (2.2∗4), and coherence—namely, the right identity triangle coherence (2.3∗8) for the first claim, and the pentagon coherence (2.3∗3) for the second. □

2.5∗18. Corollary. By induction on e and the previous lemma,

$$\mathfrak{S}\downarrow_e^{\mathrm{CommSq}_{\mathfrak{c}}(\_)} =_{\mathrm{CommSq}_{\mathfrak{c}}(X')} \mathfrak{S} \,\square\, \mathrm{idd}(e^{-1})$$

for every $\mathfrak{S} : \mathrm{CommSq}_{\mathfrak{c}}(X)$ and $e : X = X'$.

2.5∗19. Definition. *Commuting squares on a cospan.* The type of commuting squares on a cospan $\mathfrak{c}$ is the total space

$$\mathrm{CommSq}(\mathfrak{c}) :\equiv \sum_{(X \,:\, \mathscr{C}_0)} \mathrm{CommSq}_{\mathfrak{c}}(X).$$

2.5∗20. Corollary. *Equality of* $\mathrm{CommSq}(\mathfrak{c})$. Suppose that $(X, \mathfrak{S})$ and $(X', \mathfrak{S}')$ are two commuting squares on a cospan $\mathfrak{c} :\equiv (f, g)$. The equality

$$(X, \mathfrak{S}) =_{\mathrm{CommSq}(\mathfrak{c})} (X', \mathfrak{S}')$$

is equivalent to

$$\sum_{(e \,:\, X = X')}, \mathfrak{S} = \mathfrak{S}' \,\square\, \mathrm{idd}(e).$$

Proof. By the equality of Σ-types and (2.5∗18). □

2.6. PULLBACKS

In this section we study the theory of pullbacks and weak pullbacks in wild categories. These simultaneously generalize set-level 1-categorical pullbacks as well as type theoretic homotopy pullbacks as presented by Avigad, Kapulkin and Lumsdaine [AKL15].[8] The latter are instantiations of pullbacks in the universe wild categories $\mathcal{U}$ (2.1∗3), in which all the associators are trivial identity witnesses.

In the general case, composition of morphisms in an arbitrary wild category is not definitionally associative, but mediated by propositional equalities which need to be taken into account.

Pullbacks in 2-coherent wild categories can also be seen as the higher type theoretic incarnation of pseudopullbacks or comma objects in (2, 1)-categories.

2.6.1. Pullbacks and weak pullbacks

2.6.1∗1. Definition. *Notions of pullback in wild categories.* Let $\mathfrak{c} :\equiv A \xrightarrow{f} C \xleftarrow{g} B$ be a cospan in $\mathscr{C}$, $P : \mathscr{C}_0$ an object of $\mathscr{C}$, and

$$\mathfrak{P} \;:\equiv\; \begin{array}{ccc} P & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & \not\!\!/\!/_{\mathfrak{p}} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

[8] See also Rijke's development in the unpublished (draft) Chapter 22 of [Rij22b].

a commuting square on $\mathfrak{c}$ with source P. Consider the family of maps obtained by specializing the precomposition map (2.5∗16) at $\mathfrak{P}$,

$$\mathfrak{P} \mathbin{\square_{\_}} \_ : \prod_{(X\,:\,\mathscr{C}_0)} \mathscr{C}(X, P) \to \mathrm{CommSq}_{\mathfrak{c}}(X).$$

We say that $\mathfrak{P}$ is a *pullback of* $\mathfrak{c}$ if $(\mathfrak{P} \mathbin{\square_{\_}} \_)$ is a family of equivalences, and a *weak pullback of* $\mathfrak{c}$ if $(\mathfrak{P} \mathbin{\square_{\_}} \_)$ is a family of retractions, or split surjections.

2.6.1∗2. *Being a pullback is a property.* For any cospan $\mathfrak{c}$ and $P : \mathscr{C}_0$, the predicate

$$\text{is-pullback}(\mathfrak{P}) :\equiv \prod_{(X\,:\,\mathscr{C}_0)} \text{is-equiv}\,(\mathfrak{P} \mathbin{\square_X} \_)$$

on commuting squares $\mathfrak{P} : \mathrm{CommSq}_{\mathfrak{c}}(P)$ is propositional. We have the subtype

$$\mathrm{Pullback}_{\mathfrak{c}}(P) :\equiv \sum_{(\mathfrak{P}\,:\,\mathrm{CommSq}_{\mathfrak{c}}(P))}, \text{is-pullback}(\mathfrak{P})$$

of $\mathrm{CommSq}_{\mathfrak{c}}(P)$, of pullbacks of $\mathfrak{c}$ with specified source P.

2.6.1∗3. *Universal property of (weak) pullbacks.* By the characterization of equality of commuting squares (2.5∗8), for each $X : \mathscr{C}_0$ and commuting square $\mathfrak{S} :\equiv (m_A, m_B, \gamma)$ on $\mathfrak{c}$ with source X, the fiber of $(\mathfrak{P} \mathbin{\square_X} \_)$ at $\mathfrak{S}$ is equivalent to

$$\begin{aligned} &\sum_{(m\,:\,\mathscr{C}(X,P))} (e_A : \pi_A \diamond m = m_A)\,(e_B : \pi_B \diamond m = m_B), \\ &\quad \alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha = (f * e_A) \cdot \gamma \cdot (g * e_B)^{-1}. \end{aligned}$$

Thus $\mathfrak{P}$ is a pullback (respectively, a weak pullback) when this type is contractible (respectively, pointed) for every X and $\mathfrak{S}$.

2.6.2. Basic properties

2.6.2∗1. Lemma. *Transpose of pullbacks.* If $\mathfrak{P}$ is a (weak) pullback then so is $\mathfrak{P}^{\mathsf{T}}$.

Proof. By a straightforward calculation

$$(\mathfrak{P}^{\mathsf{T}} \mathbin{\square_X} \_) = (\_^{\mathsf{T}}) \circ (\mathfrak{P} \mathbin{\square_X} \_)$$

for all $X : \mathscr{C}_0$. Since $\_^{\mathsf{T}}$ is an equivalence (2.5∗11), $(\mathfrak{P}^{\mathsf{T}} \mathbin{\square_X} \_)$ is an equivalence (respectively, a retraction) when $(\mathfrak{P} \mathbin{\square_X} \_)$ is. □

2.6.2∗2. Lemma. *Precomposing (weak) pullbacks with equivalences.* Suppose that $\mathscr{C}$ is a 2-coherent wild category, $e : \mathscr{C}(P', P)$ is a $\mathscr{C}$-equivalence, and that $\mathfrak{P} : \mathrm{CommSq}_{\mathfrak{c}}(P)$ is a commuting square with source P in $\mathscr{C}$. Then $\mathfrak{P}$ is a (weak) pullback if and only if $\mathfrak{P} \mathbin{\square} e$ is a (weak) pullback.

Proof. For any $X : \mathscr{C}_0$, the triangle

$$\begin{array}{ccc} \mathscr{C}(X, P') & \xrightarrow{\;e \diamond \_\;} & \mathscr{C}(X, P) \\ & {\scriptstyle (\mathfrak{P} \,\square\, e) \,\square\, \_} \searrow \quad \swarrow {\scriptstyle \mathfrak{P} \,\square\, \_} & \\ & \mathrm{CommSq}_{\mathfrak{c}}(X) & \end{array}$$

commutes by (2.5∗17). Since $(e \diamond \_)$ is an equivalence (2.4∗2), one of the remaining sides is an equivalence (respectively, retraction) exactly when the other is. □

2.6.2∗3. Corollary. Suppose that $\mathscr{C}$ is a 2-coherent wild category. If

$$\mathfrak{P} \;:\equiv\; \begin{array}{ccc} P & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & \overset{\mathfrak{p}}{\not\!\!\!/\!\!/} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

is a pullback in $\mathscr{C}$, $e : \mathscr{C}(P', P)$ is a $\mathscr{C}$-equivalence and $m : \mathscr{C}(P', A)$ is a morphism such that

$$\begin{array}{ccc} P' & \xrightarrow[\sim]{e} & P \\ & {\scriptstyle m}\searrow \;\overset{\gamma}{\not\!\!\!/\!\!/}\; \swarrow{\scriptstyle \pi_A} & \\ & A & \end{array}$$

commutes, then the square

$$(m, \pi_B \diamond e, (f * \gamma) \cdot \alpha^{-1} \cdot (\mathfrak{p} * e) \cdot \alpha) : \mathrm{CommSq}_{(f,g)}(P'),$$

given by the canonical pasting of γ and $\mathfrak{P}$

$$\begin{array}{ccccc} P' & \xrightarrow{e} & P & \xrightarrow{\pi_B} & B \\ {\scriptstyle m}\downarrow & \overset{\gamma}{\not=} & \swarrow{\scriptstyle \pi_A} \quad \overset{\mathfrak{p}}{\not\!\!\!/\!\!/} & & \downarrow{\scriptstyle g} \\ A & & \xrightarrow[f]{} & & C \end{array},$$

is a pullback.

Proof. By (2.5∗8) the pasting is equal to $\mathfrak{P} \,\square\, e$, which is a pullback by (2.6.2∗2). □

2.6.2∗4. Lemma. *Identity pullbacks.* If $\mathscr{C}$ is a wild category with triangle coherators, then for all $A, B : \mathscr{C}_0$ and $f : \mathscr{C}(A, B)$ the commuting square

$$\mathfrak{I}_f \;:\equiv\; \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \mathrm{id}}\downarrow & \overset{\rho\cdot\lambda^{-1}}{\not\!\!\!/\!\!/} & \downarrow{\scriptstyle \mathrm{id}} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback.

Proof. We claim that for any $X : \mathscr{C}_0$,

$$\mathfrak{I}_f \,\square_X\, \_ : \mathscr{C}(X, A) \to \mathrm{CommSq}_{(f,\mathrm{id})}(X)$$

has inverse equivalence fst. It is immediate to see that fst is a retraction of $(\mathfrak{I}_f \,\square_X\, \_)$. We show that it is a section of the same, i.e. that

$$\mathfrak{I}_f \,\square\, g \;\equiv\; (\mathrm{id} \diamond g, f \diamond g, \alpha^{-1} \cdot ((\rho \cdot \lambda^{-1}) * g) \cdot \alpha) \;=\; (g, h, \gamma)$$

for any $(g, h, \gamma) : \mathrm{CommSq}_{(f,\mathrm{id})}(X)$.

To this end we use the characterization of equality for commuting squares (2.5∗8), and observe that there are equalities

$$\lambda : \mathrm{id} \diamond g = g,$$
$$\gamma \cdot \lambda : f \diamond g = h$$

for which the equation

$$\alpha^{-1} \cdot ((\rho \cdot \lambda^{-1}) * g) \cdot \alpha = (f * \lambda) \cdot \gamma \cdot (\mathrm{id} * (\gamma \cdot \lambda)^{-1})$$

holds by the algebra of whiskering (2.2∗4), the triangle coherators (2.3∗2) and (2.3∗8), and the lemma (2.3∗10). □

2.6.3. Pullback pasting

2.6.3∗1. The following lemma is inspired by the proof of [AKL15, Proposition 4.1.11] and used in the proof of the pullback pasting lemma (2.6.3∗5).

2.6.3∗2. Lemma. *Pasting maps of (weak) pullbacks.* If $\mathfrak{P}$ is a pullback (respectively, a weak pullback) in any wild category $\mathscr{C}$, then the pasting map $\bar{\bar{\mathfrak{P}}}$ of (2.5∗14) is a family of equivalences (respectively, retractions).

Proof. Let $\mathfrak{P} :\equiv (g', j, \mathfrak{p})$ be a commuting square on (k, g) as in (2.5∗14). For any $A : \mathscr{C}_0$, $f : \mathscr{C}(A, B)$ and $X : \mathscr{C}_0$, the fiber of $\bar{\bar{\mathfrak{P}}}_{A,f,X}$ at

$$\mathfrak{X} \;:\equiv\; \begin{array}{ccc} X & \xrightarrow{m_A} & A \\ {\scriptstyle m_{C'}}\big\downarrow & \not\Rightarrow_{\xi} & \big\downarrow{\scriptstyle g \diamond f} \\ C' & \xrightarrow[k]{} & C \end{array}$$

is equivalent to the Σ-type

$$\begin{gathered} \sum (m : \mathscr{C}(X, B'))\, (i : \mathscr{C}(X, A))\, (\gamma : j \diamond m = f \diamond i), \\ (e_{C'} : g' \diamond m = m_{C'}) \times (e_A : i = m_A) \\ \times\, (\alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha \cdot (g * \gamma) \cdot \alpha^{-1} = (k * e_{C'}) \cdot \xi \cdot (g \diamond f * e_A)^{-1}), \end{gathered}$$

using the characterization of equality (2.5∗8). Contracting the singleton formed by the components i and e_A, this is equivalent to

$$\begin{gathered} \sum (m : \mathscr{C}(X, B'))(e_{C'} : g' \diamond m = m_{C'})(e_B : j \diamond m = f \diamond m_A), \\ \alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha = (k * e_{C'}) \cdot (\xi \cdot \alpha) \cdot (g * e_B)^{-1}. \end{gathered}$$

But this type is also the fiber of the precomposition map $(\mathfrak{P} \square_X \_)$ at the commuting square

$$\begin{array}{ccc} X & \xrightarrow{f \diamond m_A} & B \\ {\scriptstyle m_{C'}}\big\downarrow & \not\Rightarrow_{\xi \cdot \alpha} & \big\downarrow{\scriptstyle g} \\ C' & \xrightarrow[k]{} & C \end{array}$$

obtained by "reparenthesizing" the diagram $\mathfrak{X}$. Thus if $\mathfrak{P}$ is a pullback (respectively, a weak pullback) then by its universal property (2.6.1∗3) the fiber $(\bar{\bar{\mathfrak{P}}}_{A,f,X})^{-1}(\mathfrak{X})$ is contractible (respectively, pointed). □

2.6.3∗3. In a 2-coherent wild category, the converse of (2.6.3∗2) holds:

2.6.3∗4. Lemma. *A pasting characterization of pullbacks.* Suppose that $\mathfrak{P}$ is a commuting square in a 2-coherent wild category $\mathscr{C}$. If the vertical pasting map $\overset{=}{\mathfrak{P}}$ is a family of equivalences (respectively, retractions) then $\mathfrak{P}$ is a pullback (respectively, a weak pullback).

Proof. Suppose that

$$\mathfrak{P} \quad \equiv \quad \begin{array}{ccc} B' & \xrightarrow{\;j\;} & B \\ {\scriptstyle g'}\big\downarrow & \not\!\!\!\nearrow_{\mathfrak{p}} & \big\downarrow{\scriptstyle g} \\ C' & \xrightarrow[\;k\;]{} & C \end{array}$$

is the commuting square as in (2.5∗14). For any $X : \mathscr{C}_0$, we claim that the map

$$\begin{aligned} \varphi : \mathscr{C}(X, B') &\to \mathrm{CommSq}_{(j,\mathrm{id}_B)}(X) \\ \varphi(m) &:\equiv (m, j \diamond m, {\lambda_{j \diamond m}}^{-1}) \end{aligned}$$

is an equivalence, and that the diagram of types and functions

$$\begin{array}{ccc} & & \mathrm{CommSq}_{(j,\mathrm{id}_B)}(X) \\ & \overset{\varphi}{\underset{\sim}{\nearrow}} & \big\downarrow{\scriptstyle \overset{=}{\mathfrak{P}}_{B,\mathrm{id}_B,X}} \\ \mathscr{C}(X, B') & & \mathrm{CommSq}_{(k,g \diamond \mathrm{id}_B)}(X) \\ & \underset{\mathfrak{P} \,\square_X\, \_}{\searrow} & {\scriptstyle \sim}\big\downarrow{\scriptstyle \psi} \\ & & \mathrm{CommSq}_{(k,g)}(X) \end{array}$$

commutes, where ψ is the equivalence $(m_{C'}, m_B, \gamma) \mapsto (m_{C'}, m_B, \gamma \cdot (\rho * m_B))$. That is, $(\mathfrak{P} \,\square_X\, \_)$ is the pre- and post-composition of $\overset{=}{\mathfrak{P}}_{B,\mathrm{id}_B,X}$ by equivalences. Thus, if $\overset{=}{\mathfrak{P}}$ is a family of equivalences then so is $(\mathfrak{P} \,\square_{\_}\, \_)$, and if $\overset{=}{\mathfrak{P}}$ is a family of retractions then so is $(\mathfrak{P} \,\square_{\_}\, \_)$.

Now, the map φ is clearly a section of $\mathrm{fst} : \mathrm{CommSq}_{(j,\mathrm{id})}(X) \to \mathscr{C}(X, B')$. We show that it's also a retraction of fst, i.e. that

$$\varphi(m_{B'}) \equiv (m_{B'}, j \diamond m_{B'}, \lambda^{-1}) = (m_{B'}, m_B, \gamma)$$

for all $m_{B'} : \mathscr{C}(X, B')$, $m_B : \mathscr{C}(X, B)$ and $\gamma : j \diamond m_{B'} = \mathrm{id} \diamond m_B$. Taking the straightforward equalities $\mathrm{refl} : m_{B'} = m_{B'}$ and $\gamma \cdot \lambda : j \diamond m_{B'} = m_B$ of the morphism parts of the commuting squares, we lastly need the equality of commutativity witnesses

$$\lambda^{-1} = (j * \mathrm{refl}) \cdot \gamma \cdot (\mathrm{id} * (\gamma \cdot \lambda)^{-1}),$$

or equivalently, that $\lambda \cdot \gamma = (\mathrm{id} * \gamma) \cdot (\mathrm{id} * \lambda)$. This holds by (2.3∗10).[9]

[9] Aside: this claim and its proof is very similar to the proof of (2.6.2∗4).

Finally, given $m : \mathscr{C}(X, B')$ we calculate that $\mathfrak{S} :\equiv (\psi \circ \bar{\mathfrak{P}}_{B,\mathrm{id}_B,X} \circ \varphi)(m)$ and $\mathfrak{S}' :\equiv \mathfrak{P} \square_X m$ are commuting squares of type $\mathrm{CommSq}_{(k,g)(X)}$ with the same morphism components $g' \diamond m$ and $j \diamond m$. The commutativity witness of $\mathfrak{S}$ is

$$\alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha \cdot (g * \lambda^{-1}) \cdot \alpha^{-1} \cdot (\rho * (j \diamond m)),$$

while that of $\mathfrak{S}'$ is

$$\alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha,$$

and these are equal since $(g * \lambda^{-1}) \cdot \alpha^{-1} \cdot (\rho * (j \diamond m)) = \mathrm{refl}$ by the triangle coherator. □

2.6.3∗5. Lemma. *Vertical pullback pasting.* Suppose we have a diagram

$$\begin{array}{ccc} A' & \xrightarrow{i} & A \\ {\scriptstyle f'}\downarrow & \not\!\!\!/\!\!/_{\mathfrak{q}} & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{j} & B \\ {\scriptstyle g'}\downarrow & \not\!\!\!/\!\!/_{\mathfrak{p}} & \downarrow{\scriptstyle g} \\ C' & \xrightarrow[k]{} & C \end{array}$$

in a wild category $\mathscr{C}$ that has pentagonators. Then if $\mathfrak{P} :\equiv (g', j, \mathfrak{p})$ is a pullback of (k, g), the commuting square $\mathfrak{Q} :\equiv (f', i, \mathfrak{q})$ is a pullback of (j, f) if and only if the canonical vertical pasting $\frac{\mathfrak{Q}}{\mathfrak{P}}$ is a pullback of $(k, g \diamond f)$.

Proof. We claim that for any $X : \mathscr{C}_0$, the triangle

$$\begin{array}{ccc} & & \mathrm{CommSq}_{(j,f)}(X) \\ & \overset{\mathfrak{Q}\square_X -}{\nearrow} & \downarrow{\scriptstyle \bar{\mathfrak{P}}_{A,f,X}} \\ \mathscr{C}(X, A') & & \\ & \underset{\frac{\mathfrak{Q}}{\mathfrak{P}}\square_X -}{\searrow} & \mathrm{CommSq}_{(k,g\diamond f)}(X) \end{array}$$

commutes. Then since $\mathfrak{P}$ is a pullback, the map $\bar{\mathfrak{P}}_{A,f,X}$ is an equivalence (2.6.3∗2), and it follows that $(\mathfrak{Q} \square \_)$ is a family of equivalences if and only if $(\frac{\mathfrak{Q}}{\mathfrak{P}} \square \_)$ is.

What remains is to construct a homotopy $(\frac{\mathfrak{Q}}{\mathfrak{P}} \square_X \_) = (\bar{\mathfrak{P}}_{A,f,X}) \circ (\mathfrak{Q} \square_X \_)$ for any X, i.e. a witness that, for any $m : \mathscr{C}(X, A')$, the commuting squares

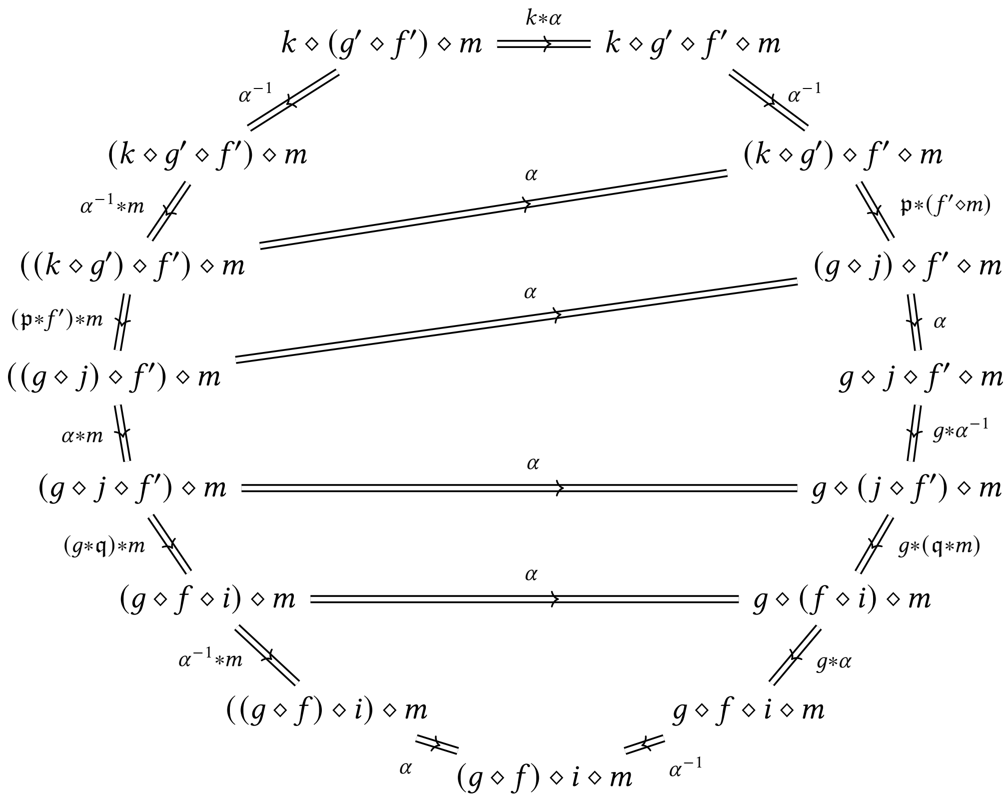
$$\frac{\mathfrak{Q}}{\mathfrak{P}} \square_X m \equiv \Big((g' \diamond f') \diamond m,\ i \diamond m,\ \alpha^{-1} \cdot \big(\tfrac{\mathfrak{q}}{\mathfrak{p}} * m\big) \cdot \alpha\Big)$$

and

$$\frac{\mathfrak{Q} \square_X m}{\mathfrak{P}} \equiv \Big(g' \diamond f' \diamond m,\ i \diamond m,\ \frac{\alpha^{-1} \cdot (\mathfrak{q} * m) \cdot \alpha}{\mathfrak{p}}\Big)$$

are equal. By (2.5∗8) together with the canonical equalities $\alpha : (g' \diamond f') \diamond m = g' \diamond f' \diamond m$ and $\mathrm{refl} : i \diamond m = i \diamond m$, it's enough to show that

$$\alpha^{-1} \cdot \big(\tfrac{\mathfrak{q}}{\mathfrak{p}} * m\big) \cdot \alpha = (k * \alpha) \cdot \frac{\alpha^{-1} \cdot (\mathfrak{q} * m) \cdot \alpha}{\mathfrak{p}}.$$



2.6.3∗5∗1. Construction of $\alpha^{-1} \cdot \left(\frac{\mathfrak{q}}{\mathfrak{p}} * m\right) \cdot \alpha = (k * \alpha) \cdot \frac{\alpha^{-1}\cdot(\mathfrak{q}*m)\cdot\alpha}{\mathfrak{p}}$.

With a little path algebra (noting (2.2∗4)) this amounts to showing commutativity of the outer boundary of Diagram 2.6.3∗5∗1.

By inserting associators α as shown in the interior of the diagram, we decompose the outer shape into a pasting of three commuting pentagons (by the pentagonators) and two commuting squares (by properties of whiskering (2.2∗4)). Thus the entire diagram commutes. □

2.6.3∗6. Corollary. *Horizontal pullback pasting.* Since the transpose of a pullback is a pullback (2.6.2∗1), by taking transposes as appropriate we deduce the more familiar horizontal version of the pullback pasting lemma.

2.6.3∗7. Definition. *Commuting squares with fixed vertices.* So far, we have considered pullbacks of a *fixed* cospan $(A, B, C, f, g) : \mathrm{Cospan}(\mathscr{C})$. We will also need to compare pullbacks on cospans with equal but definitionally distinct legs (f, g) and (f', g'). To this end, we index the type of commuting squares over their vertices, and consider the following type family indexed by $X, A, B, C : \mathscr{C}_0$,

$$\mathrm{CommSq}(X, A, B, C) :\equiv \textstyle\sum_{(f\,:\,\mathscr{C}(A,\,C))\,(g\,:\,\mathscr{C}(B,\,C))} \mathrm{CommSq}_{(f,g)}(X).$$

That is, $\mathrm{CommSq}(X, A, B, C)$ is the total space of $\mathrm{CommSq}_{(A,B,C,f,g)}(X)$ indexed over $f : \mathscr{C}(A, C)$ and $g : \mathscr{C}(B, C)$.

2.6.3∗8. Fix vertices $P, A, B, C : \mathscr{C}_0$, and suppose $(f, g, \mathfrak{P})$ and $(f', g', \mathfrak{P}')$ are equal commuting squares on P, A, B, C. By transport in the family of propositions

$$\begin{aligned} \mathrm{CommSq}(P, A, B, C) &\to \mathscr{U} \\ (f, g, \mathfrak{P}) &\mapsto \text{is-pullback}(\mathfrak{P}), \end{aligned}$$

one of these squares is a pullback exactly when the other is.

2.6.3∗9. We characterize the identity type of $\mathrm{CommSq}(X, A, B, C)$ using the following version of Rijke's structure identity principle [Rij22a, Theorem 11.6.2], which may be understood as a dependent version of the fundamental theorem of identity types.

2.6.3∗10. Theorem. *Structure identity principle.* Suppose that A is a type pointed at $a : A$, and $B : A \to \mathscr{U}$ is a type family over A, pointed at $b : B(a)$. Then for any type family

$$R : \prod\,(x : A)\ a = x \to B(x) \to \mathscr{U}$$

pointed at $r : R(a, \mathrm{refl}_a, b)$, the canonical family of maps indexed over x and y

$$\prod\,(x : A)\,(y : B(x))\ (a, b) = (x, y) \to \Sigma\,(p : a = x),\ R(x, p, y)$$

is a family of equivalences if and only if the total space $\Sigma\,(B(a))\,R(a, \mathrm{refl}_a)$ is contractible.

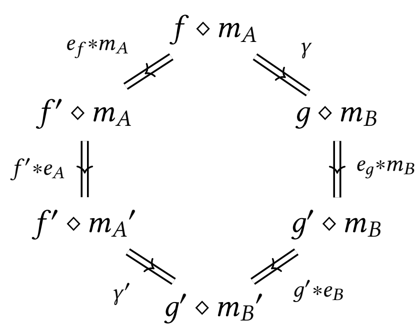
2.6.3∗11. Lemma. *Equality of* $\mathrm{CommSq}(X, A, B, C)$*.* Let

$$(f, g, \mathfrak{S}),\ (f', g', \mathfrak{S}') : \mathrm{CommSq}(X, A, B, C)$$

be commuting squares with vertices $X, A, B, C : \mathscr{C}_0$, where $\mathfrak{S} \equiv (m_A, m_B, \gamma)$ and $\mathfrak{S}' \equiv (m_A{}', m_B{}', \gamma')$. The equality $(f, g, \mathfrak{S}) = (f', g', \mathfrak{S}')$ is equivalent to

$$\begin{aligned} &\sum\,(e_f : f = f')\,(e_g : g = g')\,(e_A : m_A = m_A{}')\,(e_B : m_B = m_B{}'), \\ &\gamma = (e_f * m_A) \cdot (f' * e_A) \cdot \gamma' \cdot (g' * e_B)^{-1} \cdot (e_g * m_B)^{-1}. \end{aligned}$$

This last component is a proof that



commutes.

Proof. Use the structure identity principle (2.6.3∗10) applied to the pointed type $(\mathscr{C}(A, C) \times \mathscr{C}(B, C),\ (f, g))$ and the type family

$$\mathrm{CommSq}_{(A,B,C,\_,\_)}(X) : \mathscr{C}(A, C) \times \mathscr{C}(B, C) \to \mathscr{U}$$

pointed at $\mathfrak{S} : \mathrm{CommSq}_{(f,g)}(X)$. We define (in curried form)

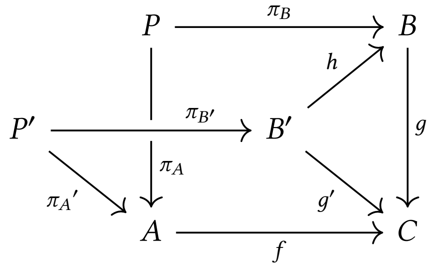
$$R : \prod\, (f' : \mathscr{C}(A, C))\, (g' : \mathscr{C}(B, C))\ (f = f') \to (g = g') \to \mathrm{CommSq}_{(f',g')}(X) \to \mathscr{U}$$

such that $R(f', g', e_f, e_g)(k_A, k_B, \delta)$ is the type

$$\begin{gathered}\sum\, (e_A : m_A = k_A)\, (e_B : m_B = k_B), \\ \gamma = (e_f * m_A) \cdot (f' * e_A) \cdot \delta \cdot (g' * e_B)^{-1} \cdot (e_g * m_B)^{-1},\end{gathered}$$

pointed at $(\mathrm{refl}_{m_A}, \mathrm{refl}_{m_B}, \mathrm{refl}_\gamma) : R(f, g, \mathrm{refl}_f, \mathrm{refl}_g, \mathfrak{S})$. Then it's enough to show that $\Sigma\, (\mathrm{CommSq}_{(f,g)}(X))\, R(f, g, \mathrm{refl}_f, \mathrm{refl}_g)$ is contractible, which we have already done in the proof of (2.5∗8). □

2.6.3∗12. Lemma. *Pullback prism.* Suppose we have a diagram



in a wild category $\mathscr{C}$ with pentagonators, such that $c : g \diamond h = g'$ is a commuting triangle, $\mathfrak{p} : f \diamond \pi_A = g \diamond \pi_B$ and $\mathfrak{p}' : f \diamond \pi_A{}' = g' \diamond \pi_{B'}$, and where the squares $\mathfrak{P} :\equiv (\pi_A, \pi_B, \mathfrak{p}) : \mathrm{CommSq}_{(f,g)}(P)$ and $\mathfrak{P}' :\equiv (\pi_A{}', \pi_{B'}, \mathfrak{p}') : \mathrm{CommSq}_{(f,g')}(P')$ are both pullbacks. Then there is a contractible type of data consisting of:

- a morphism $m : \mathscr{C}(P', P)$,
- equalities $e : \pi_A \diamond m = \pi_A{}'$ and $\mathfrak{q} : \pi_B \diamond m = h \diamond \pi_{B'}$ completing the boundary of the prism, and
- an equality 3-cell η filling the volume of the completed prism.

Even more, the top face $(m, \pi_{B'}, \mathfrak{q})$ of the completed prism is a pullback of (π_B, h).

Proof. From the universal property of $\mathfrak{P}$ we get m, e, $\mathfrak{q}$ and the equality

$$\eta : \alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha = (f * e) \cdot \mathfrak{p}' \cdot (c^{-1} * \pi_{B'}) \cdot \alpha \cdot (g * \mathfrak{q})^{-1}$$

as the center of contraction of the fiber of $(\mathfrak{P} \mathbin{\square_{P'}} \_)$ at the commuting square

$$\mathfrak{S} :\equiv \left(\pi_A{}',\ h \diamond \pi_{B'},\ \mathfrak{p}' \cdot (c^{-1} * \pi_{B'}) \cdot \alpha\right)$$

on (f, g). Let $\mathfrak{Q} :\equiv (m, \pi_{B'}, \mathfrak{q})$; by η and (2.6.3∗11) it follows that $(f, g \diamond h, \frac{\mathfrak{Q}}{\mathfrak{P}}) = (f, g', \mathfrak{P}')$ are equal commuting squares on P', A, B', C. Since $\mathfrak{P}'$ is a pullback, by (2.6.3∗8) so is $\frac{\mathfrak{Q}}{\mathfrak{P}}$, and by pullback pasting (2.6.3∗5) so too is $\mathfrak{Q}$. □

2.6.4. The truncation level of pullbacks

2.6.4∗1. Definition. *Pullbacks on a cospan.* Finally, consider the type of pullbacks on a cospan $\mathfrak{c}$,

$$\mathrm{Pullback}(\mathfrak{c}) :\equiv \sum_{(\mathfrak{P}\,:\,\mathrm{CommSq}(\mathfrak{c}))} \text{is-pullback}(\mathfrak{P}).$$

2.6.4∗2. *Equality of* Pullback($\mathfrak{c}$). Since is-pullback is a family of propositions (2.6.1∗2), equality of pullbacks on a cospan $\mathfrak{c}$ is equivalent to equality of their underlying commuting squares.

We therefore often simply denote elements of Pullback($\mathfrak{c}$) by their commuting squares, eliding the propositional witnesses that they are pullbacks.

2.6.4∗3. Pullback($\mathfrak{c}$) *is a set in set-level categories.* If $\mathscr{C}$ is set-level then Pullback($\mathfrak{c}$) is a set for any cospan $\mathfrak{c}$. This follows from the definition of CommSq($\mathfrak{c}$) (2.5∗19), and the facts (2.5∗5) and (2.6.1∗2) that $\mathrm{CommSq}_{\mathfrak{c}}$ and is-pullback are families of sets and propositions, respectively.

2.6.4∗4. Lemma. Pullback($\mathfrak{c}$) *is a proposition in univalent 2-coherent wild categories.* If $\mathscr{C}$ is a univalent 2-coherent wild category, then Pullback($\mathfrak{c}$) is a proposition for any cospan $\mathfrak{c}$ in $\mathscr{C}$.

Proof. Suppose that $(P, \mathfrak{P})$ and $(P', \mathfrak{P}')$ are elements of Pullback($\mathfrak{c}$). Then $(\mathfrak{P} \mathbin{\square} \_)$ and $(\mathfrak{P}' \mathbin{\square} \_)$ are equivalences, and from the centers of contraction of $(\mathfrak{P} \mathbin{\square} \_)^{-1}(\mathfrak{P}')$ and $(\mathfrak{P}' \mathbin{\square} \_)^{-1}(\mathfrak{P})$ we get $m : \mathscr{C}(P, P')$ and $m' : \mathscr{C}(P', P)$ such that

$$e : \mathfrak{P}' \mathbin{\square} m = \mathfrak{P} \quad \text{and} \quad e' : \mathfrak{P} \mathbin{\square} m' = \mathfrak{P}'.$$

Furthermore,

$$\mathfrak{P} \mathbin{\square} (m' \diamond m) \;=\; (\mathfrak{P} \mathbin{\square} m') \mathbin{\square} m \;=\; \mathfrak{P}' \mathbin{\square} m \;=\; \mathfrak{P}$$

by (2.5∗17), and so $m' \diamond m = \mathrm{id}_P$ by contractibility of $(\mathfrak{P} \mathbin{\square} \_)^{-1}(\mathfrak{P})$ and (2.5∗17) again.

By a similar argument $m \diamond m' = \mathrm{id}_{P'}$, and so $m : \mathscr{C}(P, P')$ is a $\mathscr{C}$-equivalence. From the univalence of $\mathscr{C}$ we now get an equality

$$\mathrm{ua}_{\mathscr{C}}(m) : P = P',$$

with

$$\mathfrak{P} \;=\; \mathfrak{P}' \mathbin{\square} m \;=\; \mathfrak{P}' \mathbin{\square} \mathrm{idd}(\mathrm{ua}_{\mathscr{C}}(m))$$

by (2.4∗10). By (2.5∗20), this proves $(P, \mathfrak{P}) = (P', \mathfrak{P}')$. □

Chapter 3

Wild Categories with Families

In this chapter we study Dybjer's categories with families [Dyb96] in homotopical type theory, while simultaneously generalizing to allow contexts to form wild categories.

3.1. WILD CATEGORIES WITH FAMILIES

3.1.1. Typed term structures

3.1.1∗1. Definition. *Typed term structures on wild categories.* Let $\mathcal{U}$ be a universe and $\mathscr{C}$ a wild category. A *typed term structure* on $\mathscr{C}$ (valued in $\mathcal{U}$) consists of the following data:

- A wild $\mathcal{U}$-valued presheaf of $\mathscr{C}$-*types* over $\mathscr{C}$, presented as a generalized algebraic theory by the components[1]

$$\begin{aligned} \mathsf{Ty} &: \mathscr{C}_0 \to \mathcal{U} \\ \_[\_]_\mathsf{T} &: \mathsf{Ty}\,\Delta \to \mathscr{C}(\Gamma, \Delta) \to \mathsf{Ty}\,\Gamma \end{aligned}$$

  and equations[1] expressing functoriality

$$\begin{aligned} [\mathrm{id}]_\mathsf{T} &: A[\mathrm{id}_\Gamma]_\mathsf{T} = A && \text{for all } A : \mathsf{Ty}\,\Gamma \\ [\diamond]_\mathsf{T} &: A[\tau \diamond \sigma]_\mathsf{T} = A[\tau]_\mathsf{T}[\sigma]_\mathsf{T} && \text{for all } A : \mathsf{Ty}\,\mathrm{E},\ \sigma : \mathscr{C}(\Gamma, \Delta),\ \tau : \mathscr{C}(\Delta, \mathrm{E}). \end{aligned}$$

[1] Implicitly quantifying over objects $\Gamma, \Delta, \mathrm{E} : \mathscr{C}_0$ as needed.

- A wild $\mathcal{U}$-valued presheaf of $\mathcal{C}$-*terms* over the (wild) category of elements of the $\mathcal{C}$-type presheaf, presented[1] by

$$\begin{aligned} \mathsf{Tm} &: (\Gamma : \mathcal{C}_0) \to \mathsf{Ty}\,\Gamma \to \mathcal{U} \\ \_[\_]_\mathsf{t} &: \mathsf{Tm}_\Delta\, A \to (\sigma : \mathcal{C}(\Gamma, \Delta)) \to \mathsf{Tm}_\Gamma\, (A[\sigma]_\mathsf{T}) \quad \text{for all} \quad A : \mathsf{Ty}\,\Delta \end{aligned}$$

and

$$\begin{aligned} [\mathrm{id}]_\mathsf{t} &: a[\mathrm{id}_\Gamma]_\mathsf{t} = a{\downarrow}^{\mathsf{Tm}_\Gamma}_{[\mathrm{id}]_\mathsf{T}^{-1}} && \text{for all} \quad A : \mathsf{Ty}\,\Gamma,\ a : \mathsf{Tm}_\Gamma\, A \\ [\diamond]_\mathsf{t} &: a[\tau \diamond \sigma]_\mathsf{t} = a[\tau]_\mathsf{t}[\sigma]_\mathsf{t}{\downarrow}^{\mathsf{Tm}_\Gamma}_{[\diamond]_\mathsf{T}^{-1}} && \text{for all} \quad A : \mathsf{Ty}\,\mathrm{E},\ a : \mathsf{Tm}_\mathrm{E}\, A \\ & && \qquad \sigma : \mathcal{C}(\Gamma, \Delta),\ \tau : \mathcal{C}(\Delta, \mathrm{E}). \end{aligned}$$

The actions $\_[\_]_\mathsf{T}$ and $\_[\_]_\mathsf{t}$ of the type and term presheaves on morphisms are called *substitution in types* and *substitution in terms*, respectively.

3.1.1∗2. We often simply denote a typed term structure on a wild category by the object parts of its component presheaves $(\mathsf{Ty}, \mathsf{Tm})$. We also frequently elide the first argument of Tm and write, for example, $\mathsf{Tm}\, A$ instead of $\mathsf{Tm}_\Gamma\, A$.

3.1.2. Equality and coherence in typed term structures

In this subsection, we record a number of notations, and simple but important observations about equivalence, equality and transport in typed term structures $(\mathsf{Ty}, \mathsf{Tm})$ on wild categories $\mathcal{C}$. We also define two coherence conditions.

3.1.2∗1. For every $\Gamma : \mathcal{C}_0$ and $A : \mathsf{Ty}\,\Gamma$, the equation $[\mathrm{id}]_\mathsf{t}$ (3.1.1∗1) implies that the function

$$\begin{aligned} \_[\mathrm{id}]_\mathsf{t} &: \mathsf{Tm}\, A \to \mathsf{Tm}\, (A[\mathrm{id}]_\mathsf{T}) \\ & a \mapsto a[\mathrm{id}]_\mathsf{t} \end{aligned}$$

is equal to transport in Tm_Γ along $[\mathrm{id}]_\mathsf{T}^{-1}$, and is hence an equivalence.

3.1.2∗2. Assume objects $\Gamma, \Delta : \mathcal{C}_0$, a $\mathcal{C}$-type $A : \mathsf{Ty}\,\Delta$, and an equality $e : \sigma = \tau$ of morphisms $\sigma, \tau : \mathcal{C}(\Gamma, \Delta)$. We write

$$[^= e]_\mathsf{T} :\equiv \mathrm{ap}\, (A[\_]_\mathsf{T})\, e$$

for the induced equality $A[\sigma]_\mathsf{T} = A[\tau]_\mathsf{T}$ By [Uni13, Lemma 2.2.2], $[^= \_]_\mathsf{T}$ respects trivial, composite and inverse equalities.

By induction on e, we also have an equality

$$[^= e]_\mathsf{t} : a[\sigma]_\mathsf{t}{\downarrow}^{\mathsf{Tm}}_{[^= e]_\mathsf{T}} = a[\tau]_\mathsf{t}$$

for any $\mathcal{C}$-term $a : \mathsf{Tm}\, A$.

3.1.2∗3. Furthermore, for any $\Gamma, \Delta : \mathcal{C}_0$, $A : \mathsf{Ty}\,\Delta$, $a : \mathsf{Tm}\,(A[\sigma]_\mathsf{T})$ and morphisms $\sigma, \tau : \mathcal{C}(\Gamma, \Delta)$ such that $e : \sigma = \tau$,

$$a{\downarrow}^{\mathsf{Tm}\,(A[\_]_\mathsf{T})}_{e} = a{\downarrow}^{\mathsf{Tm}}_{[^= e]_\mathsf{T}}$$

by [Uni13, Lemma 2.3.10].

3.1.2∗4. Suppose that $A, A' : \mathsf{Ty}\,\Delta$ are $\mathscr{C}$-types such that $e : A = A'$. For any $\sigma : \mathscr{C}(\Gamma, \Delta)$, we write

$$e[\sigma]_{\mathsf{T}} :\equiv \mathsf{ap}\,(\_[\sigma]_{\mathsf{T}})\,e$$

for the induced equality $A[\sigma]_{\mathsf{T}} = A'[\sigma]_{\mathsf{T}}$.

Similarly, if $a, a' : \mathsf{Tm}\,A$ with $e : a = a'$, we write

$$e[\sigma]_{\mathsf{t}} :\equiv \mathsf{ap}\,(\_[\sigma]_{\mathsf{t}})\,e$$

for the induced equality $a[\sigma]_{\mathsf{t}} = a'[\sigma]_{\mathsf{t}}$.

3.1.2∗5. *Substitution in transported terms.* If $e : A =_{\mathsf{Ty}\,\Delta} A'$ as in (3.1.2∗4), then for any $a : \mathsf{Tm}\,A$ and morphism $\sigma : \mathscr{C}(\Gamma, \Delta)$,

$$(a \downarrow_{e}^{\mathsf{Tm}})[\sigma]_{\mathsf{t}} = a[\sigma]_{\mathsf{t}} \downarrow_{e[\sigma]_{\mathsf{T}}}^{\mathsf{Tm}}$$

by induction on e.

3.1.2∗6. Lemma. $[\diamond]_{\mathsf{T}}$ *is a natural isomorphism.* Suppose that $\sigma, \sigma' : \mathscr{C}(\Gamma, \Delta)$ are morphisms such that $e : \sigma = \sigma'$. By induction on e, we have that the square

$$\begin{array}{ccc} A[\tau \diamond \sigma]_{\mathsf{T}} & \overset{[\diamond]_{\mathsf{T}}}{\Longrightarrow} & A[\tau]_{\mathsf{T}}[\sigma]_{\mathsf{T}} \\ {\scriptstyle [{}^{=}\tau * e]_{\mathsf{T}}}\Downarrow & & \Downarrow{\scriptstyle [{}^{=}e]_{\mathsf{T}}} \\ A[\tau \diamond \sigma']_{\mathsf{T}} & \underset{[\diamond]_{\mathsf{T}}}{\Longrightarrow} & A[\tau]_{\mathsf{T}}[\sigma']_{\mathsf{T}} \end{array}$$

canonically commutes for all $A : \mathsf{Ty}\,\mathrm{E}$ and $\tau : \mathscr{C}(\Delta, \mathrm{E})$, and also that

$$\begin{array}{ccc} A[\sigma \diamond \varrho]_{\mathsf{T}} & \overset{[\diamond]_{\mathsf{T}}}{\Longrightarrow} & A[\sigma]_{\mathsf{T}}[\varrho]_{\mathsf{T}} \\ {\scriptstyle [{}^{=}e * \varrho]_{\mathsf{T}}}\Downarrow & & \Downarrow{\scriptstyle [{}^{=}e]_{\mathsf{T}}[\varrho]_{\mathsf{T}}} \\ A[\sigma' \diamond \varrho]_{\mathsf{T}} & \underset{[\diamond]_{\mathsf{T}}}{\Longrightarrow} & A[\sigma']_{\mathsf{T}}[\varrho]_{\mathsf{T}} \end{array}$$

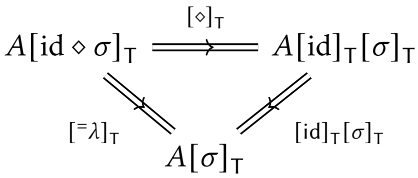
canonically commutes for all $A : \mathsf{Ty}\,\Delta$ and $\varrho : \mathscr{C}(\mathrm{B}, \Gamma)$.

3.1.2∗7. The following two definitions (3.1.2∗8) and (3.1.2∗9) are analogous to the conditions for a pseudofunctor between weak (2, 1)-categories.[2] In the cases that $\mathscr{C}$ is 2-coherent, they improve the wild presheaf Ty of a typed term structure on $\mathscr{C}$ to what might be called a *wild weak (2, 1)-presheaf*.

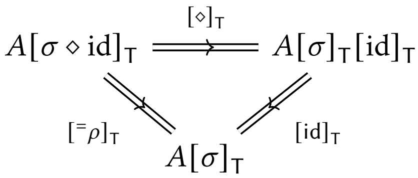
3.1.2∗8. Definition. *Type triangulators.* A wild cwf $\mathscr{C}$ is said to have *triangulators for substitution in types*, or *type triangulators*, if for all substitutions $\sigma : \mathscr{C}(\Gamma, \Delta)$

[2] See e.g. [JY21, Section 4.1].

and $\mathscr{C}$-types $A : \mathsf{Ty}\,\Delta$ the following triangles commute:



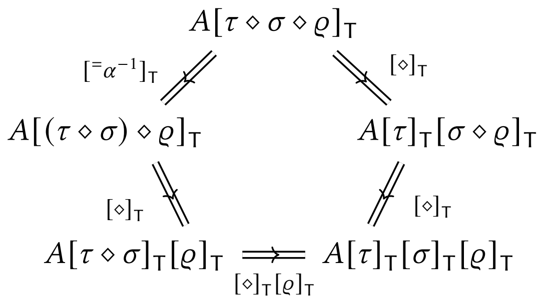
and



3.1.2∗9. Definition. *Type pentagonators.* A wild cwf $\mathscr{C}$ has *pentagonators for substitution in types*, or *type pentagonators*, if for all contexts and substitutions

$$\Gamma \xrightarrow{\varrho} \Delta \xrightarrow{\sigma} \mathrm{E} \xrightarrow{\tau} \mathrm{Z}$$

and $\mathscr{C}$-types $A : \mathsf{Ty}\,\mathrm{Z}$, the following pentagon commutes:



3.1.3. Context extension structures

3.1.3∗1. Definition. *Context extension structures.* Assume a typed term structure $(\mathsf{Ty}, \mathsf{Tm})$ on a wild category $\mathscr{C}$. A *context extension structure* on $(\mathscr{C}, \mathsf{Ty}, \mathsf{Tm})$ is given[3] by the following components

$$\begin{aligned}
_._ &: (\Gamma : \mathscr{C}_0) \to \mathsf{Ty}\,\Gamma \to \mathscr{C}_0 \\
\mathsf{p} &: (A : \mathsf{Ty}\,\Gamma) \to \mathscr{C}(\Gamma.A, \Gamma) \\
\mathsf{q} &: (A : \mathsf{Ty}\,\Gamma) \to \mathsf{Tm}_{\Gamma.A}\,(A[\mathsf{p}_A]_{\mathsf{T}}) \\
_,_ &: (\sigma : \mathscr{C}(\Gamma, \Delta)) \to \mathsf{Tm}_{\Gamma}\,(A[\sigma]_{\mathsf{T}}) \to \mathscr{C}(\Gamma, \Delta.A) \quad \text{for all} \quad A : \mathsf{Ty}\,\Delta
\end{aligned}$$

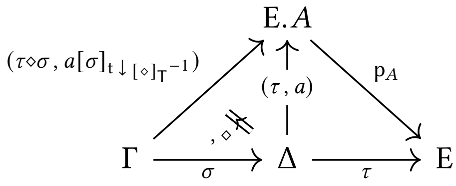
and equations (note (3.1.2∗2))

$$\begin{aligned}
\mathsf{p}\beta &: \mathsf{p}_A \diamond (\sigma, a) = \sigma && \text{and} \\
\mathsf{q}\beta &: \mathsf{q}_A[\sigma, a]_{\mathsf{t}} = a\downarrow^{\mathsf{Tm}}_{[^{=}\mathsf{p}\beta]_{\mathsf{T}}{}^{-1}\cdot[\diamond]_{\mathsf{T}}} && \text{for all} \quad \sigma : \mathscr{C}(\Gamma, \Delta), \\
& && \qquad A : \mathsf{Ty}\,\Delta,\ a : \mathsf{Tm}_{\Gamma}\,(A[\sigma]_{\mathsf{T}}) \\
,\eta &: (\mathsf{p}_A, \mathsf{q}_A) = \mathrm{id}_{\Gamma.A} && \text{for all} \quad A : \mathsf{Ty}\,\Gamma \\
,\diamond &: (\tau, a) \diamond \sigma = (\tau \diamond \sigma, a[\sigma]_{\mathsf{t}}\downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}{}^{-1}}) && \text{for all} \quad \sigma : \mathscr{C}(\Gamma, \Delta),\ \tau : \mathscr{C}(\Delta, \mathrm{E}), \\
& && \qquad A : \mathsf{Ty}\,\mathrm{E},\ a : \mathsf{Tm}_{\Delta}\,(A[\tau]_{\mathsf{T}}).
\end{aligned}$$

We call p the *display map*, and q the *generic term* of the context extension structure.

[3]Again, implicitly generalizing over $\Gamma, \Delta, \mathrm{E} : \mathscr{C}_0$ as needed.

3.1.3∗2. In particular, the equation $(,\diamond)$ (3.1.3∗1) witnesses the commutativity of the inner left triangle in the diagram



3.1.3∗3. We sometimes elide the argument $A : \mathsf{Ty}\,\Gamma$ to the display map p and the generic term q of a context extension structure. When we need to be concise, we denote the display map $\Gamma.A \xrightarrow{\mathrm{p}_A} \Gamma$ by $\Gamma.A \twoheadrightarrow \Gamma$.

3.1.4. Cwf structures and wild cwfs

3.1.4∗1. Definition. *Cwf structures on wild categories.* If $\mathscr{C}$ is a wild category, a *cwf structure* on $\mathscr{C}$ consists of:

- a terminal object $\blacklozenge : \mathscr{C}_0$,
- a typed term structure $(\mathsf{Ty}, \mathsf{Tm})$ on $\mathscr{C}$, and
- a context extension structure on $(\mathscr{C}, \mathsf{Ty}, \mathsf{Tm})$,

modeling the structural rules of a Martin-Löf type theory over $\mathscr{C}$.

3.1.4∗2. Definition. *Wild categories with families.* A *wild category with families (wild cwf)* is a wild category $\mathscr{C}$ together with a cwf structure on $\mathscr{C}$. In this case, we call $\mathscr{C}$ the *category of contexts* of the wild cwf, its objects *contexts*, and its morphisms *substitutions*.

3.1.4∗3. We usually denote a wild cwf simply by its category of contexts.

3.1.4∗4. The presentation of a category with families as a generalized algebraic theory (GAT) already appears in Dybjer [Dyb96], in a metatheory with equality satisfying UIP, and thus assuming a set-level 1-category of contexts. We simply observe that these definitions carry over verbatim to the case where the underlying category of contexts is wild. In this latter case the specified notion of cwf is more algebraic, since it has explicitly chosen witnesses of equations which are not necessarily propositional.

3.1.4∗5. Example. *Set-level cwfs.* Every *set-level cwf*—one with a set-level category of contexts, and whose presheaves of types and terms are valued in sets—is immediately a wild cwf.

3.1.4∗6. Example. *Universe cwfs.* If a universe wild category $\mathcal{U}$ has Σ and unit types that satisfy the η-rule, then it supports a canonical wild cwf structure given as follows.

- The terminal context $\blacklozenge$ is the unit type $\mathbb{1} : \mathcal{U}$.

- The typed term structure is as follows. $\mathcal{U}$-types in context Δ are Δ-indexed type families

$$\begin{aligned}\mathsf{Ty} &: \mathcal{U} \to \mathcal{U}^{+}\\ \mathsf{Ty}\,\Delta &:\equiv \Delta \to \mathcal{U},\end{aligned}$$

  while $\mathcal{U}$-terms $a : \mathsf{Tm}_{\Delta}\,A$ are sections of $A : \mathsf{Ty}\,\Delta$

$$\mathsf{Tm}_{\Delta}\,A :\equiv \Pi\,\Delta\,A.$$

  Substitution of $\sigma : \mathcal{U}(\Gamma, \Delta)$ in $\mathcal{U}$-types $A : \mathsf{Ty}\,\Delta$ and $\mathcal{U}$-terms a is given by precomposition

$$\begin{aligned}A[\sigma]_{\mathsf{T}} &:\equiv A \circ \sigma,\\ a[\sigma]_{\mathsf{t}} &:\equiv a \circ \sigma.\end{aligned}$$

  This action is definitionally functorial—that is, $[\mathrm{id}]_{\mathsf{T}}$, $[\circ]_{\mathsf{T}}$, $[\mathrm{id}]_{\mathsf{t}}$ and $[\circ]_{\mathsf{t}}$ are all families of trivial identities.
- The context extension structure is given by dependent pairing. The extended context $\Delta.A$ is $\Sigma\,\Delta\,A$, and p and q are the functions fst and snd respectively. For $\sigma : \mathcal{U}(\Gamma, \Delta)$ and $t : \mathsf{Tm}_{\Gamma}\,(A \circ \sigma)$, the extended substitution $(\sigma, t) : \mathcal{U}(\Gamma, \Sigma\,\Delta\,A)$ is given by

$$(\sigma, t)(\gamma) :\equiv (\sigma(\gamma), t(\gamma)).$$

  Again, the equations for context extension structures hold definitionally. In particular, the η-rule for Σ-types is used for , η.

We refer to the resulting wild cwf as the *universe cwf*. Variations of this canonical cwf structure also appear throughout the literature as the “standard model”.

3.1.4∗7. Example. *Subuniverse cwfs and the 1-cwf of sets.* The construction of the typed term and context extension structures of (3.1.4∗6) works equally well for any *sub*universe wild category (2.1∗4) that has a terminal object and is closed under Σ-types with η.

In particular, the 1-cwf[4] of sets $\mathsf{Set}_{\mathcal{U}}$ is a subuniverse cwf of $\mathcal{U}$.

3.1.4∗8. Definition. *Univalent wild cwfs.* A wild cwf $\mathscr{C}$ is called *univalent* if its category of contexts is univalent.

3.1.4∗9. Examples. Any subuniverse of a univalent universe $\mathcal{U}$ yields a univalent cwf. In particular, $\mathsf{Set}_{\mathcal{U}}$ and $\mathcal{U}$ are univalent.

3.2. BASIC STRUCTURAL PROPERTIES

For the rest of this chapter we assume that $\mathscr{C}$ is a wild cwf.

[4] i.e. wild cwf with a 1-category of contexts, and for which Ty and Tm are set-valued.

3.2∗1. Lemma. *Substitutions into extended contexts are pairs.* Let $\Gamma, \Delta : \mathscr{C}_0$ be contexts, and $A : \mathsf{Ty}\,\Delta$ a $\mathscr{C}$-type. There is an equivalence

$$\mathscr{C}(\Gamma,\, \Delta.A) \underset{(\sigma, a) \,\leftarrow\!\shortmid\, (\sigma, a)}{\overset{\sigma \mapsto (\mathsf{p}_A \diamond \sigma,\ \mathsf{q}_A[\sigma]_{\mathsf{t}} \downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}{}^{-1}})}{\simeq}} \sum_{(\sigma : \mathscr{C}(\Gamma,\, \Delta))} \mathsf{Tm}\,(A[\sigma]_{\mathsf{T}}),$$

where the reverse function sends a pair (σ, a) to the extended substitution (σ, a) given by the context extension structure (3.1.3∗1).

Proof. For one composition, it's enough to show that for all $\sigma : \mathscr{C}(\Gamma,\, \Delta.A)$,

$$(\mathsf{p}_A \diamond \sigma, \mathsf{q}_A[\sigma]_{\mathsf{t}} \downarrow_{[\diamond]_{\mathsf{T}}{}^{-1}}) \overset{,\,\diamond^{-1}}{=\!=} (\mathsf{p}_A, \mathsf{q}_A) \diamond \sigma \overset{,\,\eta * \sigma}{=\!=} \mathrm{id} \diamond \sigma \overset{\lambda}{=} \sigma.$$

For the other, we show the equality of pairs

$$(\mathsf{p}_A \diamond (\sigma, a), \mathsf{q}_A[\sigma, a]_{\mathsf{t}} \downarrow_{[\diamond]_{\mathsf{T}}{}^{-1}}) = (\sigma, a).$$

Equality of the first components is given by $\mathsf{p}\beta$ (3.1.3∗1), and for the second components we have that

$$\begin{aligned} & \mathsf{q}_A[\sigma, a]_{\mathsf{t}} \downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}{}^{-1}} \downarrow^{\mathsf{Tm}\,(A[\_]_{\mathsf{T}})}_{\mathsf{p}\beta} \\ =\ & \mathsf{q}_A[\sigma, a]_{\mathsf{t}} \downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}{}^{-1} \cdot [{=}\mathsf{p}\beta]_{\mathsf{T}}} && \text{(by (3.1.2∗3) and (1.2∗5))} \\ =\ & a && \text{(by } \mathsf{p}\beta \text{ and properties of transport).} \end{aligned}$$

□

3.2∗2. Corollary. *Elimination principle for* $\mathscr{C}(\Gamma,\, \Delta.A)$. By the previous lemma (3.2∗1), to construct a section of a family of types P over $\mathscr{C}(\Gamma,\, \Delta.A)$, it's enough to give an element of $P((\sigma, a))$ for every $\sigma : \mathscr{C}(\Gamma,\, \Delta)$ and $a : \mathsf{Tm}\,(A[\sigma]_{\mathsf{T}})$.

3.2∗3. Corollary. *Equality of substitutions into extended contexts.* If σ and τ are substitutions from Γ to $\Delta.A$, then by the previous lemma (3.2∗1), the fact that equivalences induce equivalent identity types [Uni13, Theorem 2.11.1], and the equality (3.1.2∗3), an equality $\sigma = \tau$ is equivalent to a pair of equalities $e : \mathsf{p} \diamond \sigma = \mathsf{p} \diamond \tau$ and $\mathsf{q}[\sigma]_{\mathsf{t}} \downarrow_{[\diamond]_{\mathsf{T}}{}^{-1} \cdot [{=}e]_{\mathsf{T}} \cdot [\diamond]_{\mathsf{T}}} = \mathsf{q}[\tau]_{\mathsf{t}}$.

An alternative but equivalent[5] formulation is the following—for substitutions of the form $(\sigma, a), (\tau, b) : \mathscr{C}(\Gamma,\, \Delta.A)$,

$$(\sigma, a) =_{\mathscr{C}(\Gamma,\, \Delta.A)} (\tau, b) \simeq \sum_{(e : \sigma = \tau)} a \downarrow^{\mathsf{Tm}}_{[{=}e]_{\mathsf{T}}} = b.$$

To see this, write φ for the forward equivalence of (3.2∗1). In the proof of (3.2∗1) we showed that the equalities of pairs $\varphi((\sigma, a)) = (\sigma, a)$ and $\varphi((\tau, b)) = (\tau, b)$ hold. Then $\varphi((\sigma, a)) = \varphi((\tau, b))$ is equivalent to the equality type of pairs $(\sigma, a) = (\tau, b)$, which by (3.1.2∗3) is equivalent to the Σ-type as claimed.

[5] Via a proof (definitionally) different from the one given here.

3.2∗4. Lemma. *Terms are sections of display maps.* For all contexts $\Gamma : \mathscr{C}_0$ and $\mathscr{C}$-types $A : \mathsf{Ty}\,\Gamma$, there is an equivalence

$$\mathsf{Tm}\,A \simeq \mathrm{Sect}(\mathrm{p}_A)$$

whose forward map sends the $\mathscr{C}$-term a to the section $(\mathrm{id}, a[\mathrm{id}]_\mathrm{t})$ of p_A, witnessed by $\mathrm{p}\beta$.

Proof. We have that

$$\begin{aligned}
&\mathrm{Sect}(\mathrm{p}_A) \\
\equiv\ &\textstyle\sum (\sigma : \mathscr{C}(\Gamma, \Gamma.A)),\ \mathrm{p}_A \diamond \sigma = \mathrm{id} \\
\simeq\ &\textstyle\sum (u : \Sigma\,(\sigma : \mathscr{C}(\Gamma, \Gamma)), \mathsf{Tm}\,(A[\sigma]_\mathrm{T})),\ \mathrm{p}_A \diamond (\mathrm{fst}\,u, \mathrm{snd}\,u) = \mathrm{id} && \text{(by (3.2∗1))} \\
\simeq\ &\textstyle\sum (u : \Sigma\,(\sigma : \mathscr{C}(\Gamma, \Gamma)), \mathsf{Tm}\,(A[\sigma]_\mathrm{T})),\ \mathrm{fst}\,u = \mathrm{id} && \text{(by p}\beta\text{ (3.1.3∗1))} \\
\simeq\ &\textstyle\sum (u : \Sigma\,(\sigma : \mathscr{C}(\Gamma, \Gamma)), \sigma = \mathrm{id}),\ \mathsf{Tm}\,(A[\mathrm{fst}\,u]_\mathrm{T}) && \text{(assoc. of }\Sigma\text{ and comm. of }\times) \\
\simeq\ &\mathsf{Tm}\,(A[\mathrm{id}]_\mathrm{T}) && \text{(contractibility of singletons)} \\
\simeq\ &\mathsf{Tm}\,A && \text{(by the inverse of the equivalence }_[\mathrm{id}]_\mathrm{t}\text{ (3.1.2∗1)).}
\end{aligned}$$

Tracing the composition of this chain of equivalences, we compute that its inverse is equal to the map

$$\begin{aligned}
\mathsf{Tm}\,A &\longrightarrow \mathrm{Sect}(\mathrm{p}_A) \\
a &\longmapsto ((\mathrm{id}, a[\mathrm{id}]_\mathrm{t}), \mathrm{p}\beta).
\end{aligned}$$ □

3.2∗5. Analogues of the properties (3.2∗1) and (3.2∗4) were already observed for set-level cwfs by Dybjer [Dyb96] and by Hofmann [Hof97] in a traditional 1-categorical setting. In that setting, these properties essentially follow from the fact that context extension structures on $(\mathscr{C}, \mathsf{Ty}, \mathsf{Tm})$ are choices of representing objects for particular presheaves on slices of $\mathscr{C}$.

It may seem slightly surprising that the fully coherent homotopical versions of the same properties hold for arbitrary, even noncoherent, wild cwfs. On the other hand, given that the equivalent natural models [Awo18] have a relatively simple axiomatization in terms of representability of pullbacks of presheaves, it is perhaps to be expected that certain consequences would carry over immediately to higher generalizations, even in the absence of higher coherence.

3.3. EQUALITY AND COHERENCE FOR CONTEXT EXTENSION

3.3∗1. We now construct and analyze in more detail a characterization

$$\mathrm{sub}^{=}_{\sigma,\tau} : \left(\textstyle\sum (e : \mathrm{p} \diamond \sigma = \mathrm{p} \diamond \tau),\ \mathrm{q}[\sigma]_\mathrm{t} \downarrow_{[\diamond]_\mathrm{T}^{-1} \cdot [^{=}e]_\mathrm{T} \cdot [\diamond]_\mathrm{T}} = \mathrm{q}[\tau]_\mathrm{t}\right) \xrightarrow{\sim} \sigma = \tau$$

of the equality of substitutions $\sigma, \tau : \mathscr{C}(\Gamma, \Delta.A)$ into extended contexts.

3.3∗2. First let σ and τ be substitutions from Γ to *arbitrary* contexts Δ, with $A : \mathsf{Ty}\,\Delta$, $a : \mathsf{Tm}\,(A[\sigma]_\mathsf{T})$ and $b : \mathsf{Tm}\,(A[\tau]_\mathsf{T})$. From the equivalence

$$\Big(\textstyle\sum\,(\sigma : \mathscr{C}(\Gamma,\,\Delta)),\ \mathsf{Tm}\,(A[\sigma]_\mathsf{T})\Big) \xrightarrow[(\_,\,\_)]{\sim} \mathscr{C}(\Gamma,\,\Delta.A)$$

of (3.2∗1), we obtain

$$(\sigma, a) =_{\Sigma\,(\mathscr{C}(\Gamma,\,\Delta.A))\,(\mathsf{Tm}\,(A[\_]_\mathsf{T}))} (\tau, b) \xrightarrow[\mathsf{ap}\,(\_,\,\_)]{\sim} (\sigma, a) =_{\mathscr{C}(\Gamma,\,\Delta.A)} (\tau, b).$$

Precomposing this with

$$\Big(\textstyle\sum\,(e : \sigma = \tau),\ a\downarrow^{\mathsf{Tm}}_{[=e]_\mathsf{T}} = b\Big) \xrightarrow[\mathrm{id}\times\varphi]{\sim} \textstyle\sum\,(e : \sigma = \tau),\ a\downarrow^{\mathsf{Tm}\,(A[\_]_\mathsf{T})}_{e} = b$$
$$\xrightarrow[\mathsf{pair}^=]{\sim} (\sigma, a) =_{\Sigma\,(\mathscr{C}(\Gamma,\,\Delta.A))\,(\mathsf{Tm}\,(A[\_]_\mathsf{T}))} (\tau, b),$$

where

$$\varphi : \textstyle\prod\,(e : \sigma = \tau)\ \Big(a\downarrow^{\mathsf{Tm}}_{[=e]_\mathsf{T}} = b \xrightarrow{\sim} a\downarrow^{\mathsf{Tm}\,(A[\_]_\mathsf{T})}_{e} = b\Big)$$

is the family of equivalences induced by (3.1.2∗3) and $\mathsf{pair}^=$ is the standard characterization of the equality of Σ-types, we get an equivalence

$$\mathsf{sub}_0^= : \Big(\textstyle\sum\,(e : \sigma = \tau),\ a\downarrow^{\mathsf{Tm}}_{[=e]_\mathsf{T}} = b\Big) \xrightarrow{\sim} (\sigma, a) =_{\mathscr{C}(\Gamma,\,\Delta.A)} (\tau, b)$$
$$\mathsf{sub}_0^=\,(e, e') :\equiv \mathsf{ap}\,(\_,\,\_)\,(\mathsf{pair}^=\,(e, \varphi_e e')).$$

3.3∗3. $\mathsf{p}\beta$ *is a natural isomorphism.* By definition, $\mathsf{sub}_0^=\,(\mathsf{refl}, \mathsf{refl}) \equiv \mathsf{refl}$. Hence for all substitutions $\sigma, \tau : \mathscr{C}(\Gamma,\,\Delta.A)$, $\mathscr{C}$-terms $a : \mathsf{Tm}\,(A[\sigma]_\mathsf{T})$ and $b : \mathsf{Tm}\,(A[\tau]_\mathsf{T})$, and equalities $e : \sigma = \tau$ and $e' : a\downarrow_{[=e]_\mathsf{T}} = b$, we have that the square

$$\begin{array}{ccc} \mathsf{p}_A \diamond (\sigma, a) & \xRightarrow{\ \mathsf{p}\beta\ } & \sigma \\ {\scriptstyle \mathsf{p}_A\,\ast\,\mathsf{sub}_0^=\,(e,e')}\Big\Downarrow & & \Big\Downarrow{\scriptstyle e} \\ \mathsf{p}_A \diamond (\tau, b) & \xRightarrow[\ \mathsf{p}\beta\ ]{} & \tau \end{array}$$

canonically commutes by induction on e and e'. Equivalently,

$$\mathsf{p}_A \ast \mathsf{sub}_0^=\,(e, e') = \mathsf{p}\beta \cdot e \cdot \mathsf{p}\beta^{-1}.$$

3.3∗4. η-*equality of substitutions.* For all $A : \mathsf{Ty}\,\Delta$ and $\sigma : \mathscr{C}(\Gamma,\,\Delta.A)$, denote by

$$\eta^{\mathsf{sub}}_\sigma : (\mathsf{p}_A \diamond \sigma, \mathsf{q}_A[\sigma]_{\mathsf{t}\downarrow[\diamond]_\mathsf{T}^{-1}}) = \sigma$$
$$\eta^{\mathsf{sub}}_\sigma :\equiv\ ,\diamond^{-1} \cdot (,\eta \ast \sigma) \cdot \lambda$$

the equality in the first part of the proof of (3.2∗1). This is an η-rule for substitutions into extended contexts.

3.3∗5. Definition. If $\sigma, \tau : \mathscr{C}(\Gamma, \Delta.A)$ are substitutions such that

$$e : \mathsf{p}_A \diamond \sigma = \mathsf{p}_A \diamond \tau$$

and

$$e' : \mathsf{q}[\sigma]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}\cdot[{=}e]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}}} = \mathsf{q}[\tau]_{\mathsf{t}},$$

define $\mathsf{sub}^{=}_{\sigma,\tau}(e, e')$ to be the composite

$$\sigma \overset{\eta^{\mathsf{sub}\,-1}_{\sigma}}{=\!=} (\mathsf{p} \diamond \sigma, \mathsf{q}[\sigma]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}}) \overset{\mathsf{sub}^{=}_{0}(e,e'')}{=\!=\!=} (\mathsf{p} \diamond \tau, \mathsf{q}[\tau]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}}) \overset{\eta^{\mathsf{sub}}_{\tau}}{=\!=} \tau,$$

where $e'' : \mathsf{q}[\sigma]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}\downarrow[{=}e]_{\mathsf{T}}} = \mathsf{q}[\tau]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}}$ is canonically constructed from e'. This definition yields an equivalence as promised in (3.3∗1), being essentially the composition of $\mathsf{sub}^{=}_{0}$ with the equivalence given by path composition with $\eta^{\mathsf{sub}\,-1}_{\sigma}$ and $\eta^{\mathsf{sub}}_{\tau}$.

3.3∗6. It is natural to ask if a β-rule holds for the first argument of $\mathsf{sub}^{=}_{\sigma,\tau}$, i.e. if, for all σ and τ, the composition

$$\Big(\sum_{(\mathsf{p}\diamond\sigma=\mathsf{p}\diamond\tau)} \mathsf{q}[\sigma]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}\cdot[{=}\_]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}}} = \mathsf{q}[\tau]_{\mathsf{t}}\Big) \xrightarrow{\mathsf{sub}^{=}_{\sigma,\tau}} \sigma = \tau \xrightarrow{\mathsf{p}\,*\,\_} \mathsf{p} \diamond \sigma = \mathsf{p} \diamond \tau$$

is equal to the first projection. By (3.3∗3), we calculate that

$$\mathsf{p} * \mathsf{sub}^{=}_{\sigma,\tau}(e, e') = (\mathsf{p} * \eta^{\mathsf{sub}\,-1}_{\sigma}) \cdot \mathsf{p}\beta \cdot e \cdot \mathsf{p}\beta^{-1} \cdot (\mathsf{p} * \eta^{\mathsf{sub}}_{\tau})$$

for all e and e'. The desire to have this expression be equal to e motivates the next definition.

3.3∗7. Definition. *Coherators for* η^{sub}. We say that a wild cwf $\mathscr{C}$ *has coherators for* η^{sub} if for all $\Gamma, \Delta : \mathscr{C}_0$, $A : \mathsf{Ty}\,\Delta$ and $\sigma : \mathscr{C}(\Gamma, \Delta.A)$, we have that

$$\mathsf{p}_A * \eta^{\mathsf{sub}}_{\sigma} = \mathsf{p}\beta$$

are equal 2-cells of type $\mathsf{p}_A \diamond (\mathsf{p}_A \diamond \sigma, \mathsf{q}_A[\sigma]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}}) = \mathsf{p}_A \diamond \sigma$.

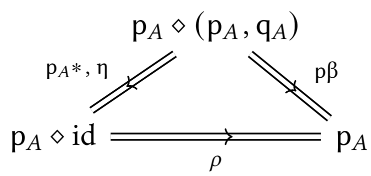
3.3∗8. Lemma. Suppose $\sigma, \tau : \mathscr{C}(\Gamma, \Delta.A)$ are equal substitutions, witnessed by $e : \mathsf{p}_A \diamond \sigma = \mathsf{p}_A \diamond \tau$ and $e' : \mathsf{q}[\sigma]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}\cdot[{=}e]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}}} = \mathsf{q}[\tau]_{\mathsf{t}}$. If $\mathscr{C}$ has coherators for η^{sub}, then

$$\mathsf{p}_A * \mathsf{sub}^{=}_{\sigma,\tau}(e, e') = e.$$

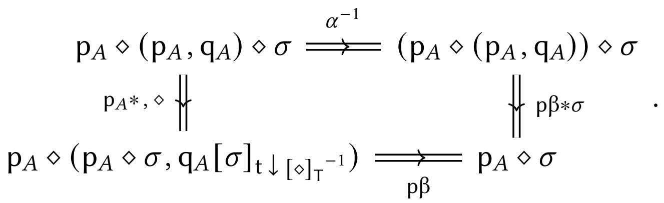
Proof. Immediate, from the calculation (3.3∗6) and the coherence assumption. □

3.3∗9. In fact, a wild cwf $\mathscr{C}$ has coherators for η^{sub} if it has triangle coherators (2.3∗2) and satisfies the following coherence condition:

3.3∗10. Definition. *Coherators for context extension.* A wild cwf $\mathscr{C}$ *has coherators for context extension* if, for all $\Gamma, \Delta : \mathscr{C}_0$, $A : \mathsf{Ty}\,\Delta$ and $\sigma : \mathscr{C}(\Gamma, \Delta.A)$, the following diagrams of equalities commute:

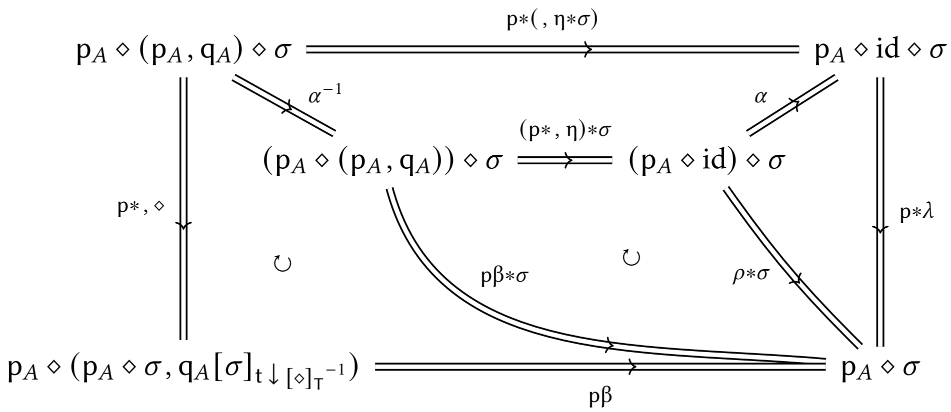


and



3.3∗11. Lemma. If a wild cwf $\mathscr{C}$ has triangle coherators as well as coherators for context extension, then it has coherators for η^{sub}.

Proof. Having coherators for η^{sub} is equivalent to having the outer boundary of Diagram 3.3∗11∗1 commute for all $\Gamma, \Delta : \mathscr{C}_0$, $A : \mathrm{Ty}\,\Delta$ and $\sigma : \mathscr{C}(\Gamma, \Delta.A)$. We show that this holds by pasting together the regions shown in the interior of the diagram, where the topmost interior square is filled by the third associativity law for whiskering (2.2∗4), the rightmost triangle by the triangle coherator, and the regions marked ↻ using the coherators for context extension. □



3.3∗11∗1. Constructing coherators for η^{sub} from coherators for context extension.

3.3∗12. Example. The universe cwfs $\mathscr{U}$ have trivial coherators for context extension.

3.3∗13. Definition. *Structurally 2-coherent wild cwfs.* We say that a wild cwf $\mathscr{C}$ is *(structurally) 2-coherent* if $\mathscr{C}$ has

- a 2-coherent wild category of contexts (2.3∗4),
- type triangulators (3.1.2∗8) and type pentagonators (3.1.2∗9), and

- coherators for context extension (3.3∗10).

3.3∗14. Presumably, the coherences for context extension structures should arise as instances of more general universal properties of sufficiently coherent representable presheaves à la a formulation via wild natural models.

3.4. ON SUBSTITUTION

3.4.1. Substitution in types

3.4.1∗1. A prominent feature of fibrational models in the categorical semantics of dependent type theory is that substitution in types satisfies the universal property of pullbacks. In (3.4.1∗2) and (3.4.1∗3), we show that the same is true for 2-coherent wild cwfs.

3.4.1∗2. Lemma. *2-coherent substitution in types is weak pullback.* Suppose that $\mathscr{C}$ is a 2-coherent wild cwf, $\sigma : \mathscr{C}(\Gamma, \Delta)$ is a substitution, and $A : \mathsf{Ty}\,\Delta$ is a $\mathscr{C}$-type. Then there is a substitution

$$\sigma^{\cdot A} :\equiv (\sigma \diamond \mathsf{p}_{A[\sigma]_\mathsf{T}}, \mathsf{q}_{A[\sigma]_\mathsf{T}} \downarrow^{\mathsf{Tm}}_{[\diamond]_\mathsf{T}^{-1}})$$

from $\Gamma.A[\sigma]_\mathsf{T}$ to $\Delta.A$, such that the following square with source $\Gamma.A[\sigma]_\mathsf{T}$

$$\mathfrak{P}_{\sigma,A} \;:\equiv\; \begin{array}{ccc} \Gamma.A[\sigma]_\mathsf{T} & \xrightarrow{\sigma^{\cdot A}} & \Delta.A \\ {\scriptstyle \mathsf{p}}\downarrow & \overset{\mathsf{p}\beta^{-1}}{/\!\!/} & \downarrow{\scriptstyle \mathsf{p}} \\ \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array}$$

is a weak pullback in $\mathscr{C}$. That is, for any $\mathrm{B} : \mathscr{C}_0$ and commuting square $\mathfrak{S} :\equiv (\tau, \varrho, \gamma)$ with source B as in

$$\begin{array}{ccc} \mathrm{B} & \xrightarrow{\varrho} & \Delta.A \\ {\scriptstyle \tau}\downarrow & \overset{\gamma}{/\!\!/} & \downarrow{\scriptstyle \mathsf{p}} \\ \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array},$$

the fiber $(\mathfrak{P}_{\sigma,A} \mathbin{\square} \_)^{-1}(\mathfrak{S})$ is pointed, i.e. there is a mediating substitution

$$\mu_{\sigma,A,\mathfrak{S}} : \mathscr{C}(\mathrm{B}, \Gamma.A[\sigma]_\mathsf{T})$$

such that

$$\theta_{\sigma,A,\mathfrak{S}} : \mathfrak{P}_{\sigma,A} \mathbin{\square} \mu_{\sigma,A,\mathfrak{S}} = \mathfrak{S}.$$

Proof. We claim that a mediating substitution is given by

$$\mu_{\sigma,A,\mathfrak{S}} :\equiv (\tau, \mathsf{q}_A[\varrho]_\mathsf{t} \downarrow^{\mathsf{Tm}}_{[\diamond]_\mathsf{T}^{-1}\cdot[{=}\gamma]_\mathsf{T}^{-1}\cdot[\diamond]_\mathsf{T}}),$$

where $\mathsf{q}_A[\varrho]_\mathsf{t} : \mathsf{Tm}\,A[\mathsf{p}_A]_\mathsf{T}[\varrho]_\mathsf{T}$ is transported in the family Tm_B along

$$A[\mathsf{p}]_\mathsf{T}[\varrho]_\mathsf{T} \xlongequal{[\diamond]_\mathsf{T}^{-1}} A[\mathsf{p} \diamond \varrho]_\mathsf{T} \xlongequal{[{=}\gamma]_\mathsf{T}^{-1}} A[\sigma \diamond \tau]_\mathsf{T} \xlongequal{[\diamond]_\mathsf{T}} A[\sigma]_\mathsf{T}[\tau]_\mathsf{T}.$$

For brevity, denote $\mu_{\sigma,A,\mathfrak{S}}$ by μ. By (2.5∗8), constructing $\theta : \mathfrak{P}_{\sigma,A} \mathbin{\square} \mu = \mathfrak{S}$ is equivalent to constructing witnesses

$$\delta : \mathsf{p}_{A[\sigma]_\mathsf{T}} \diamond \mu = \tau$$

and

$$\epsilon : \sigma^{\cdot A} \diamond \mu = \varrho$$

such that

$$\alpha^{-1} \cdot (\mathsf{p}\beta^{-1} * \mu) \cdot \alpha = (\sigma * \delta) \cdot \gamma \cdot (\mathsf{p}_A * \epsilon)^{-1}.$$

Let $\delta :\equiv \mathsf{p}\beta$. Using the equivalence $\mathsf{sub}^{=}_{\_,\_}$ defined at (3.3∗5), we define

$$\epsilon :\equiv \mathsf{sub}^{=}_{(\sigma^{\cdot A} \diamond \mu),\varrho}(\epsilon_0, \epsilon_1),$$

where

$$\epsilon_0 : \mathsf{p}_A \diamond \sigma^{\cdot A} \diamond \mu = \mathsf{p}_A \diamond \varrho$$

and

$$\epsilon_1 : \mathsf{q}_A[\sigma^{\cdot A} \diamond \mu]_\mathsf{t} \downarrow^{\mathsf{Tm}}_{[\diamond]_\mathsf{T}^{-1} \cdot [^{=}\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T}} = \mathsf{q}_A[\varrho]_\mathsf{t}$$

are the 2-cells[6] constructed as follows.

First, let ϵ_0 be the concatenation of equalities

$$\mathsf{p}_A \diamond \sigma^{\cdot A} \diamond \mu \overset{\alpha^{-1} \cdot (\mathsf{p}\beta * \mu) \cdot \alpha}{=\!=\!=\!=\!=} \sigma \diamond \mathsf{p}_{A[\sigma]_\mathsf{T}} \diamond \mu \overset{\sigma * \mathsf{p}\beta}{=\!=\!=} \sigma \diamond \tau \overset{\gamma}{=\!=} \mathsf{p}_A \diamond \varrho.$$

Now calculate that

$$\begin{aligned}
& \mathsf{q}_A[\sigma^{\cdot A} \diamond \mu]_\mathsf{t} \\
= \;& \mathsf{q}_A[\sigma^{\cdot A}]_\mathsf{t}[\mu]_\mathsf{t} \downarrow_{[\diamond]_\mathsf{T}^{-1}} && \text{(by } [\diamond]_\mathsf{t}\text{)} \\
\equiv \;& \mathsf{q}_A[\sigma \diamond \mathsf{p}_{A[\sigma]_\mathsf{T}}, \mathsf{q}_{A[\sigma]_\mathsf{T}} \downarrow_{[\diamond]_\mathsf{T}^{-1}}]_\mathsf{t}[\mu]_\mathsf{t} \downarrow_{[\diamond]_\mathsf{T}^{-1}} \\
= \;& (\mathsf{q}_{A[\sigma]_\mathsf{T}} \downarrow_{[\diamond]_\mathsf{T}^{-1} \cdot [^{=}\mathsf{p}\beta]_\mathsf{T}^{-1} \cdot [\diamond]_\mathsf{T}})[\mu]_\mathsf{t} \downarrow_{[\diamond]_\mathsf{T}^{-1}} && \text{(by } \mathsf{q}\beta\text{)} \\
= \;& \mathsf{q}_{A[\sigma]_\mathsf{T}}[\mu]_\mathsf{t} \downarrow_{([\diamond]_\mathsf{T}^{-1} \cdot [^{=}\mathsf{p}\beta]_\mathsf{T}^{-1} \cdot [\diamond]_\mathsf{T})[\mu]_\mathsf{T} \cdot [\diamond]_\mathsf{T}^{-1}} && \text{(by (3.1.2}*\text{5))} \\
= \;& \mathsf{q}_A[\varrho]_\mathsf{t} \downarrow_e && \text{(by } \mathsf{q}\beta\text{)}
\end{aligned}$$

where the transports are all in Tm_B, and e is the composition

$$e :\equiv [\diamond]_\mathsf{T}^{-1} \cdot [^{=}\gamma]_\mathsf{T}^{-1} \cdot [\diamond]_\mathsf{T} \cdot [^{=}\mathsf{p}\beta]_\mathsf{T}^{-1} \cdot [\diamond]_\mathsf{T} \cdot ([\diamond]_\mathsf{T}^{-1} \cdot [^{=}\mathsf{p}\beta]_\mathsf{T}^{-1} \cdot [\diamond]_\mathsf{T})[\mu]_\mathsf{T} \cdot [\diamond]_\mathsf{T}^{-1}.$$

So to construct ϵ_1, we may as well show that

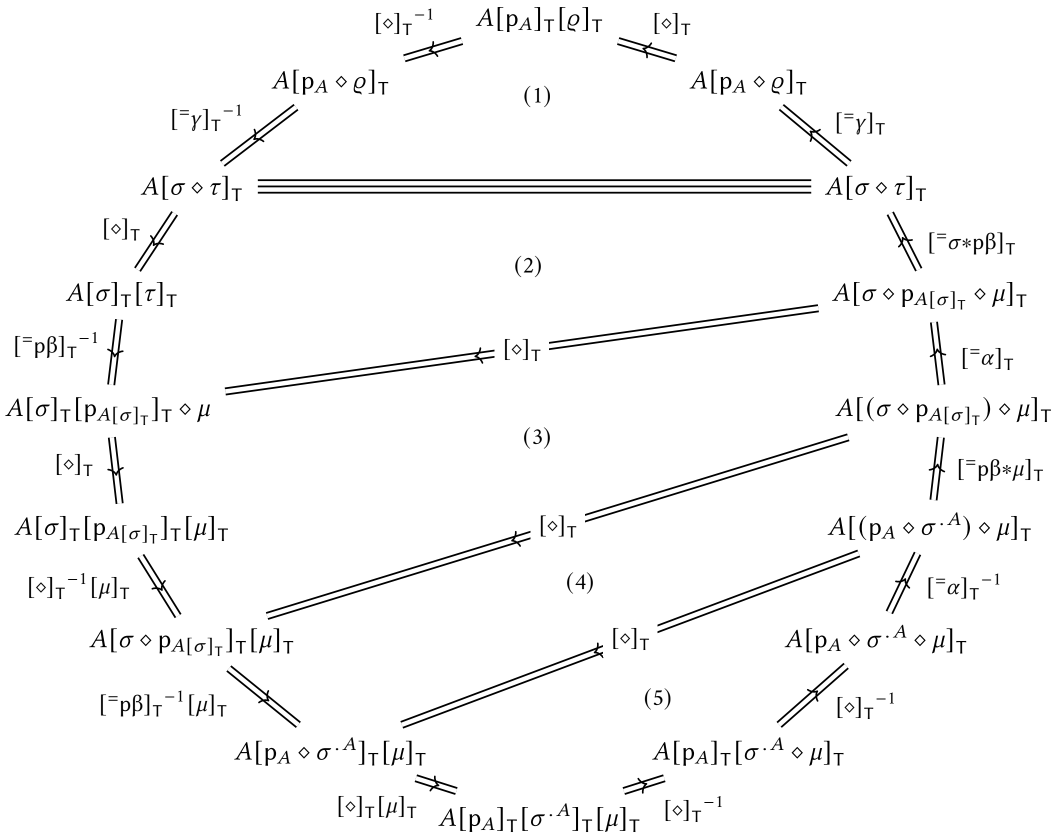
$$\mathsf{q}_A[\varrho]_\mathsf{t} \downarrow_{e \cdot [\diamond]_\mathsf{T}^{-1} \cdot [^{=}\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T}} = \mathsf{q}_A[\varrho]_\mathsf{t}.$$

We do this by showing that the equality

$$\widetilde{e} :\equiv e \cdot [\diamond]_\mathsf{T}^{-1} \cdot [^{=}\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T}$$

is in fact equal to the trivial identity. Some path algebra shows that $\widetilde{e}$ is equal to the outer boundary of Diagram 3.4.1∗2∗1. This boundary commutes, since we can fill the interior of the diagram with the following commuting regions:

[6] Since $\mathscr{C}$-terms correspond to display map morphisms in $\mathscr{C}$ (3.2∗4), we are justified in calling equalities between terms 2-cells.



3.4.1∗2∗1. $e \cdot [\diamond]_\mathsf{T}{}^{-1} \cdot [^=\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T} = \mathrm{refl}.$

- (1), which commutes straightforwardly,
- (2) and (4), which commute by (3.1.2∗6), and
- (3) and (5), which are filled by the type pentagonators.

This shows that $\widetilde{e} = \mathrm{refl}$, which completes the proof ϵ_1 that

$$\mathrm{q}_A[\sigma^{\cdot A} \diamond \mu]_\mathsf{t} \downarrow^{\mathsf{Tm}}_{[\diamond]_\mathsf{T}{}^{-1} \cdot [^=\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T}} = \mathrm{q}_A[\varrho]_\mathsf{t} \downarrow_{\widetilde{e}} = \mathrm{q}_A[\varrho]_\mathsf{t},$$

and thus also the proof $\epsilon :\equiv \mathrm{sub}^{=}_{(\sigma^{\cdot A} \diamond \mu),\, \varrho}(\epsilon_0, \epsilon_1)$ that

$$\sigma^{\cdot A} \diamond \mu = \varrho.$$

Finally, what remains is to show that

$$\alpha^{-1} \cdot (\mathrm{p}\beta^{-1} * \mu) \cdot \alpha = (\sigma * \delta) \cdot \gamma \cdot (\mathrm{p}_A * \epsilon)^{-1}.$$

But by (3.3∗8) and (3.3∗11) we have that $(\mathrm{p}_A * \epsilon)^{-1} = \epsilon_0^{-1}$ on the right hand side, and the equality then follows by calculation. □

3.4.1∗3. Theorem. *2-coherent substitution in types is pullback.* The weak pullbacks $\mathfrak{P}_{\sigma, A}$ of the previous lemma (3.4.1∗2) are pullbacks.

Proof. By (3.4.1∗2) we have that, for any $\mathrm{B} : \mathscr{C}_0$, the map

$$\mu_{\sigma,A} : \mathrm{CommSq}_{(\sigma,\mathrm{p}_A)}(\mathrm{B}) \longrightarrow \mathscr{C}(\mathrm{B}, \Gamma.A[\sigma]_\mathsf{T})$$
$$\mu_{\sigma,A}((\tau, \varrho, \gamma)) :\equiv (\tau, \mathrm{q}_A[\varrho]_{\mathrm{t}} \downarrow^{\mathsf{Tm}}_{[\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\gamma]_\mathsf{T}{}^{-1} \cdot [\diamond]_\mathsf{T}})$$

is a section of the precomposition map $(\mathfrak{P}_{\sigma,A} \,\square_{\mathrm{B}}\, \_)$. We show that it's a retraction of the same, and therefore that $(\mathfrak{P}_{\sigma,A} \,\square_{\_}\, \_)$ is a family of equivalences.

That is, for $m : \mathscr{C}(\mathrm{B}, \Gamma.A[\sigma]_\mathsf{T})$, we want the equality of substitutions

$$\mu_{\sigma,A}(\mathfrak{P}_{\sigma,A} \,\square\, m) = m.$$

By (3.2∗3) and a calculation similar to the one in the proof of (3.4.1∗2) we have that

$$\mu_{\sigma,A}(\mathfrak{P}_{\sigma,A} \,\square\, m) = (\mathrm{p} \diamond m, \mathrm{q}_{A[\sigma]_\mathsf{T}}[m]_{\mathrm{t}} \downarrow_e),$$

where

$$e :\equiv ([\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\mathrm{p}\beta]_\mathsf{T}{}^{-1} \cdot [\diamond]_\mathsf{T})[m]_\mathsf{T} \cdot [\diamond]_\mathsf{T}{}^{-1} \cdot [\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\alpha^{-1} \cdot (\mathrm{p}\beta * m) \cdot \alpha]_\mathsf{T} \cdot [\diamond]_\mathsf{T}.$$

On the other hand,

$$m = (\mathrm{p} \diamond m, \mathrm{q}_{A[\sigma]_\mathsf{T}}[m]_{\mathrm{t}} \downarrow_{[\diamond]_\mathsf{T}{}^{-1}})$$

by (3.2∗1), and thus by (3.2∗3) again it's enough to show that $e = [\diamond]_\mathsf{T}{}^{-1}$.

By path algebra this amounts to showing the commutativity of a diagram of equalities that looks like the one formed by regions (3), (4) and (5) of Diagram 3.4.1∗2∗1, but where we replace μ with m. Commutativity of this diagram then follows as in the proof of (3.4.1∗2), i.e. by (3.1.2∗6) and type pentagonators. □

3.4.1∗4. *On the uniqueness of pullback data in set-level and universe cwfs.* Assume $\Gamma, \Delta : \mathscr{C}_0$, and let $\mathfrak{c} :\equiv \Gamma \xrightarrow{\sigma} \Delta \xleftarrow{\mathrm{p}_A} \Delta.A$ be a cospan in $\mathscr{C}_0$.

If $\mathscr{C}$ is a universe cwf then $\mathrm{Pullback}(\mathfrak{c})$ is contractible, since it is a proposition (2.6.4∗4) inhabited by $(\Gamma.A[\sigma]_\mathsf{T}, \mathfrak{P}_{\sigma,A})$. On the other hand, if $\mathscr{C}$ only has a precategory of contexts then $\mathrm{Pullback}(\mathfrak{c})$ may have elements with isomorphic, but propositionally distinct, source objects.

The situation is reversed if we *fix* the source and edges of a pullback square on $\mathfrak{c}$. In particular, if $\mathscr{C}$ is set-level then $\mathrm{p}\beta^{-1}$ is the unique 2-cell filling the interior of the pullback square $\mathfrak{P}_{\sigma,A}$. This is *not* the case in universe cwfs, by the following result.

3.4.1∗5. Lemma. Let $\Gamma : \mathscr{U}_0$ be a context in a universe cwf $\mathscr{U}$, and assume that for all $\Delta : \mathscr{U}_0$, $\sigma : \mathscr{U}(\Gamma, \Delta)$ and $A : \mathrm{Ty}\,\Delta$, the square

$$\begin{array}{ccc} \Gamma.A[\sigma]_\mathsf{T} & \xrightarrow{\sigma \cdot A} & \Delta.A \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array}$$

admits a unique 2-filler $\gamma : \sigma \diamond \mathrm{p}_{A[\sigma]_\mathsf{T}} = \mathrm{p}_A \diamond \sigma^{\cdot A}$. Then Γ is a set.

Proof. Unfolding definitions and choosing $\Delta :\equiv \Gamma$ and $\sigma :\equiv \mathrm{id}_\Gamma$, the premise gives us that for all $A : \Gamma \to \mathscr{U}$, the square

$$\begin{array}{ccc} \Sigma\,\Gamma\,A & \xrightarrow{\mathrm{id}} & \Sigma\,\Gamma\,A \\ {\scriptstyle \mathrm{fst}}\downarrow & & \downarrow{\scriptstyle \mathrm{fst}} \\ \Gamma & \xrightarrow[\mathrm{id}]{} & \Gamma \end{array}$$

has a unique filler; in other words, that the type $\mathrm{fst} =_{(\Sigma\,\Gamma\,A \to \Gamma)} \mathrm{fst}$ is contractible.

To see that Γ is a set, we show that the type $\gamma_0 = \gamma_0$ is contractible for an arbitrary $\gamma_0 : \Gamma$. To this end consider the type family $A(\gamma) :\equiv \gamma = \gamma_0$. Then the composition of canonical equivalences

$$(\Sigma\,\Gamma\,A \to \Gamma) \xrightarrow{\sim} (\mathbb{1} \to \Gamma) \xrightarrow{\sim} \Gamma$$

sends fst to γ_0, and thus induces an equivalence

$$(\mathrm{fst} = \mathrm{fst}) \simeq (\gamma_0 = \gamma_0).$$

□

3.4.2. Split comprehension

3.4.2∗1. Definition. *Cleavings.* Let $\mathscr{C}$ be a 2-coherent wild cwf. We define a *cleaving of* $\mathscr{C}$ to be an assignment

$$\mathfrak{cl} : \prod\,(\Gamma, \Delta : \mathscr{C}_0)\,(\sigma : \mathscr{C}(\Gamma, \Delta))\,(A : \mathsf{Ty}\,\Delta)\ \mathrm{Pullback}_{(\sigma, \mathrm{p}_A)}(\Gamma.A[\sigma]_\mathsf{T})$$

of pullbacks

$$\mathfrak{cl}_{\Gamma,\Delta}(\sigma, A) \quad \equiv \quad \begin{array}{ccc} \Gamma.A[\sigma]_\mathsf{T} & \xrightarrow{\ell_{\sigma,A}} & \Delta.A \\ \downarrow & \mathfrak{p}_{\sigma,A} & \downarrow \\ \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array}$$

to every cospan of the form $\Gamma \xrightarrow{\sigma} \Delta \xleftarrow{\mathrm{p}_A} \Delta.A$. We call the component $\ell_{\sigma,A}$ of the pullback $\mathfrak{cl}_{\Gamma,\Delta}(\sigma, A)$ the *chosen lift of σ at A*.

3.4.2∗2. Definition. *The type substitution cleaving.* Every 2-coherent wild cwf has a cleaving

$$\mathfrak{cl}_{\Gamma,\Delta}(\sigma, A) :\equiv \mathfrak{P}_{\sigma,A}$$

given by the type substitution pullback squares of (3.4.1∗3). In particular, $\sigma^{\cdot A}$ is the chosen lift of a substitution $\sigma : \mathscr{C}(\Gamma, \Delta)$ at $A : \mathsf{Ty}\,\Delta$.

3.4.2∗3. Lemma. Let $e : A = A'$ be an equality of $\mathscr{C}$-types $A, A' : \mathsf{Ty}\,\Gamma$. Then

$$\mathrm{idd}(\mathrm{ap}\,(\Gamma.\_)\ e) = (\mathrm{p}_A, \mathrm{q}_A{\downarrow}^{\mathsf{Tm}}_{e[\mathrm{p}_A]_\mathsf{T}}).$$

Proof. By induction on e it's enough to show that $\mathrm{idd}(\mathrm{refl}_{\Gamma.A}) = (\mathrm{p}_A, \mathrm{q}_A)$, which holds by , η (3.1.3∗1). □

3.4.2∗4. Lemma. The type substitution cleaving of any 2-coherent wild cwf satisfies

$$\mathrm{id}^{\cdot A} = \mathrm{idd}(\mathrm{ap}\,(\Gamma.\_)\,[\mathrm{id}]_{\mathrm{T}})$$

for any $\Gamma : \mathscr{C}_0$ and $A : \mathrm{Ty}\,\Gamma$.

Proof. By (3.4.2∗3) it's enough to show that $\mathrm{id}^{\cdot A} = (\mathrm{p}_{A[\mathrm{id}]_{\mathrm{T}}}, \mathrm{q}_{A[\mathrm{id}]_{\mathrm{T}}} \downarrow [\mathrm{id}]_{\mathrm{T}}[\mathrm{p}_{A[\mathrm{id}]_{\mathrm{T}}}]_{\mathrm{T}})$, which holds by (3.2∗3) and the left type triangulator (3.1.2∗8) of a 2-coherent wild cwf. □

3.4.2∗5. Definition. *Split cleavings of 2-coherent wild cwfs.* A cleaving $\mathfrak{cl}$ of a 2-coherent wild cwf $\mathscr{C}$ is called *split* if for all substitutions

$$\mathrm{B} \xrightarrow{\tau} \Gamma \xrightarrow{\sigma} \Delta$$

and $\mathscr{C}$-types $A : \mathrm{Ty}\,\Delta$, the equality type

$$\big(\mathrm{B}.A[\sigma \diamond \tau]_{\mathrm{T}},\ \mathfrak{cl}_{\mathrm{B},\Delta}(\sigma \diamond \tau, A)\big) = \big(\mathrm{B}.A[\sigma]_{\mathrm{T}}[\tau]_{\mathrm{T}},\ \mathfrak{cl}_{\mathrm{B},\Gamma}(\tau, A[\sigma]_{\mathrm{T}}) \mid \mathfrak{cl}_{\Gamma,\Delta}(\sigma, A)\big)$$

of the pullbacks

$$\mathfrak{cl}_{\mathrm{B},\Delta}(\sigma \diamond \tau, A) \equiv \begin{array}{ccc} \mathrm{B}.A[\sigma \diamond \tau]_{\mathrm{T}} & \xrightarrow{\ell_{\sigma\diamond\tau,A}} & \Delta.A \\ \downarrow & \mathfrak{p}_{\sigma\diamond\tau,A} & \downarrow \\ \mathrm{B} & \xrightarrow[\sigma\diamond\tau]{} & \Delta \end{array}$$

and

$$\mathfrak{cl}_{\mathrm{B},\Gamma}(\tau, A[\sigma]_{\mathrm{T}}) \mid \mathfrak{cl}_{\Gamma,\Delta}(\sigma, A) \equiv \begin{array}{ccccc} \mathrm{B}.A[\sigma]_{\mathrm{T}}[\tau]_{\mathrm{T}} & \xrightarrow{\ell_{\tau,A[\sigma]_{\mathrm{T}}}} & \Gamma.A[\sigma]_{\mathrm{T}} & \xrightarrow{\ell_{\sigma,A}} & \Delta.A \\ \downarrow & \mathfrak{p}_{\tau,A[\sigma]_{\mathrm{T}}} & \downarrow & \mathfrak{p}_{\sigma,A} & \downarrow \\ \mathrm{B} & \xrightarrow[\tau]{} & \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array}$$

on $\mathrm{B} \xrightarrow{\sigma\diamond\tau} \Delta \xleftarrow{\mathrm{p}_A} \Delta.A$ is contractible.

3.4.2∗6. A split cleaving of a 2-coherent internal cwf, in our sense, may be considered a higher version of a splitting of a full comprehension category. As is to be expected, set-level internal cwfs have split type substitution cleavings. We now show that the type substitution cleavings of *univalent* 2-coherent wild cwfs are also split.

3.4.2∗7. Lemma. Suppose $\mathscr{C}$ is a 2-coherent wild cwf. For all substitutions $\mathrm{B} \xrightarrow{\tau} \Gamma \xrightarrow{\sigma} \Delta$ and $\mathscr{C}$-types $A : \mathrm{Ty}\,\Delta$,

$$(\mathrm{B}.A[\sigma \diamond \tau]_{\mathrm{T}},\ \mathfrak{P}_{\sigma\diamond\tau,A}) = (\mathrm{B}.A[\sigma]_{\mathrm{T}}[\tau]_{\mathrm{T}},\ \mathfrak{P}_{\tau,A[\sigma]_{\mathrm{T}}} \mid \mathfrak{P}_{\sigma,A}).$$

Proof. By (2.5∗20) it's enough to give

$$e : \mathrm{B}.A[\sigma \diamond \tau]_{\mathrm{T}} = \mathrm{B}.A[\sigma]_{\mathrm{T}}[\tau]_{\mathrm{T}}$$

such that

$$\mathfrak{P}_{\sigma\diamond\tau, A} = (\mathfrak{P}_{\tau, A[\sigma]_\mathsf{T}} \mid \mathfrak{P}_{\sigma, A}) \,\square\, \mathrm{idd}(e).$$

Take

$$e :\equiv \mathrm{ap}\,(\mathsf{B}.\_)\,[\diamond]_\mathsf{T},$$

then by (3.4.2∗3) we may as well show that

$$\mathfrak{P}_{\sigma\diamond\tau, A} = (\mathfrak{P}_{\tau, A[\sigma]_\mathsf{T}} \mid \mathfrak{P}_{\sigma, A}) \,\square\, (\mathsf{p}, \mathsf{q}_{\downarrow\,[\diamond]_\mathsf{T}[\mathsf{p}]_\mathsf{T}}),$$

or, equivalently, give three equalities

$$\delta : \mathsf{p}_{A[\sigma\diamond\tau]_\mathsf{T}} = \mathsf{p}_{A[\sigma]_\mathsf{T}[\tau]_\mathsf{T}} \diamond (\mathsf{p}, \mathsf{q}_{\downarrow\,[\diamond]_\mathsf{T}[\mathsf{p}]_\mathsf{T}}),$$
$$\epsilon : (\sigma \diamond \tau)^{\cdot A} = (\sigma^{\cdot A} \diamond \tau^{\cdot A[\sigma]_\mathsf{T}}) \diamond (\mathsf{p}, \mathsf{q}_{\downarrow\,[\diamond]_\mathsf{T}[\mathsf{p}]_\mathsf{T}})$$

and

$$\eta : \mathfrak{p}_{\sigma\diamond\tau,A} = \big((\sigma\diamond\tau) * \delta\big) \cdot \alpha^{-1} \cdot \big((\mathfrak{p}_{\tau,A[\sigma]_\mathsf{T}} \mid \mathfrak{p}_{\sigma,A}) * (\mathsf{p}, \mathsf{q}_{\downarrow\,[\diamond]_\mathsf{T}[\mathsf{p}]_\mathsf{T}})\big) \cdot \alpha \cdot (\mathsf{p}_A * \epsilon^{-1}).$$

Take $\delta :\equiv \mathsf{p}\beta^{-1}$. We will define $\epsilon :\equiv \mathrm{sub}^{=}(\epsilon_0, \epsilon_1)$, where $\mathrm{sub}^{=}$ is the equivalence defined at (3.3∗5), and where we seek equalities

$$\epsilon_0 : \mathsf{p} \diamond (\sigma\diamond\tau)^{\cdot A} = \mathsf{p} \diamond (\sigma^{\cdot A} \diamond \tau^{\cdot A[\sigma]_\mathsf{T}}) \diamond (\mathsf{p}, \mathsf{q}_{\downarrow\,[\diamond]_\mathsf{T}[\mathsf{p}]_\mathsf{T}})$$

and

$$\epsilon_1 : \mathsf{q}[(\sigma\diamond\tau)^{\cdot A}]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}{}^{-1}\cdot[^{=}\epsilon_0]_\mathsf{T}\cdot[\diamond]_\mathsf{T}} = \mathsf{q}[(\sigma^{\cdot A} \diamond \tau^{\cdot A[\sigma]_\mathsf{T}}) \diamond (\mathsf{p}, \mathsf{q}_{\downarrow\,[\diamond]_\mathsf{T}[\mathsf{p}]_\mathsf{T}})]_\mathsf{t}.$$

Now, from (3.3∗8) we have that

$$\mathsf{p}_A * \epsilon^{-1} = (\mathsf{p}_A * \epsilon)^{-1} = \epsilon_0^{-1},$$

and by rearranging the type of η we may take

$$\epsilon_0 :\equiv \mathfrak{p}_{\sigma\diamond\tau,A}{}^{-1} \cdot \big((\sigma\diamond\tau) * \delta\big) \cdot \alpha^{-1} \cdot \big((\mathfrak{p}_{\tau,A[\sigma]_\mathsf{T}} \mid \mathfrak{p}_{\sigma,A}) * (\mathsf{p}, \mathsf{q}_{\downarrow\,[\diamond]_\mathsf{T}[\mathsf{p}]_\mathsf{T}})\big) \cdot \alpha.$$

What remains, then, is to construct ϵ_1. By applying $\mathsf{q}\beta$ and $[\diamond]_\mathsf{t}$ to reduce the generic terms on the left and right, we calculate that its type is equivalent to

$$\mathsf{q}_{A[\sigma\diamond\tau]_\mathsf{t}\,\downarrow^{\mathsf{Tm}\,\mathsf{B}}_{e_1}} = \mathsf{q}_{A[\sigma\diamond\tau]_\mathsf{t}\,\downarrow^{\mathsf{Tm}\,\mathsf{B}}_{e_2}},$$

where the left and right hand sides are transported, respectively, over equalities

$$e_1 :\equiv [\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\mathsf{p}\beta^{-1}]_\mathsf{T} \cdot [^{=}\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T}$$

and

$$e_2 :\equiv [\diamond]_\mathsf{T}[\mathsf{p}]_\mathsf{T} \cdot [^{=}\mathsf{p}\beta^{-1}]_\mathsf{T} \cdot [\diamond]_\mathsf{T} \cdot \widetilde{e}\,[\mathsf{p}, \mathsf{q}_{\downarrow\,[\diamond]_\mathsf{T}[\mathsf{p}]_\mathsf{T}}]_\mathsf{T} \cdot [\diamond]_\mathsf{T}{}^{-1},$$

and where

$$\widetilde{e} :\equiv [\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\mathsf{p}\beta^{-1}]_\mathsf{T} \cdot [\diamond]_\mathsf{T} \cdot \big([\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\mathsf{p}\beta^{-1}]_\mathsf{T} \cdot [\diamond]_\mathsf{T}\big)[\tau^{\cdot A[\sigma]_\mathsf{T}}]_\mathsf{T} \cdot [\diamond]_\mathsf{T}{}^{-1}.$$

Similarly to the approach used in the proof of (3.4.1∗2), it's now enough to show that $e_1 = e_2$. This amounts to giving a filling of Diagram 3.4.2∗7∗1, which we divide into three regions filled with coherence cells as shown in Diagrams 3.4.2∗7∗2 and Diagram 3.4.2∗7∗3. □

3.4.2∗8. Corollary. *Split comprehension for set-level and univalent cwfs.* The type substitution cleaving $\mathfrak{P}_{\sigma,A}$ of any set-level or univalent 2-coherent wild cwf $\mathscr{C}$ is split.

Proof. By (3.4.2∗4), the lift of an identity substitution is equal to the dependent identity substitution over $[\mathrm{id}]_\mathsf{T}$. By (2.6.4∗3), (2.6.4∗4) and (3.4.2∗7), the equality

$$(\mathrm{B}.A[\sigma \diamond \tau]_\mathsf{T},\ \mathfrak{P}_{\sigma\diamond\tau,A}) = (\mathrm{B}.A[\sigma]_\mathsf{T}[\tau]_\mathsf{T},\ \mathfrak{P}_{\tau,A[\sigma]_\mathsf{T}} \mid \mathfrak{P}_{\sigma,A})$$

of pullbacks is an inhabited proposition for all appropriately typed σ, τ and A when $\mathscr{C}$ is set-level, or 2-coherently univalent. □

3.4.2∗9. The pasting proofs for (3.4.1∗2) and (3.4.2∗7) involve filling a diagram whose boundary B is given by concatenating two complicated chains of equalities e_1 and ${e_2}^{-1}$. In each of these cases, the cells of the pasting have boundaries that are either completely disjoint from B, or else include edges from both e_1 and e_2. It is not clear to me if the proofs can be simplified by simplifying e_1 or e_2 separately, or if there is a higher-level proof that avoids explicitly calculating the boundary and filling with basic coherences.

3.4.3. Nested substitutions

The following property of substitution can be proved from the pullback property of type substitution (3.4.1∗2). It is needed if, for instance, one wants to define Σ-types internal to wild cwfs.

3.4.3∗1. Lemma. *Nested substitutions are pullbacks.* Let $\Gamma, \Delta : \mathscr{C}_0$ be contexts in a 2-coherent wild cwf $\mathscr{C}$, $\sigma : \mathscr{C}(\Gamma, \Delta)$ a $\mathscr{C}$-substitution, and $a : \mathsf{Tm}\,A$. Then the sections

$$\bar{a} :\equiv (\mathrm{id}, a[\mathrm{id}]_\mathsf{t}) \quad\text{and}\quad \overline{a[\sigma]_\mathsf{t}} :\equiv (\mathrm{id}, a[\sigma]_\mathsf{t}[\mathrm{id}]_\mathsf{t})$$

of, respectively, $\mathsf{p}_A : \mathscr{C}(\Delta.A, \Delta)$ and $\mathsf{p}_{A[\sigma]_\mathsf{T}} : \mathscr{C}(\Gamma.A[\sigma]_\mathsf{T}, \Gamma)$ form the vertical sides of a pullback square

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ \sigma\ } & \Delta \\ {\scriptstyle \overline{a[\sigma]_\mathsf{t}}}\downarrow & & \downarrow{\scriptstyle \bar{a}} \\ \Gamma.A[\sigma]_\mathsf{T} & \xrightarrow[\ \sigma\cdot A\ ]{} & \Delta.A \end{array} .$$

Proof. Consider the commuting square

$$\mathfrak{P} :\equiv \begin{array}{ccc} \Gamma & \xrightarrow{\ \sigma\ } & \Delta \\ {\scriptstyle \mathsf{p}\diamond\overline{a[\sigma]_\mathsf{t}}}\downarrow & \not\nearrow_{\mathfrak{p}} & \downarrow{\scriptstyle \mathsf{p}\diamond\bar{a}} \\ \Gamma & \xrightarrow[\ \sigma\ ]{} & \Delta \end{array}$$

where $\mathfrak{p} :\equiv (\sigma * \mathsf{p}\beta) \cdot \rho \cdot \lambda^{-1} \cdot (\mathsf{p}\beta^{-1} * \sigma)$. It is straightforward to show that $\mathfrak{P}$ is equal to $\mathfrak{I}_\sigma$ (2.6.2∗4) (as elements of $\mathrm{CommSq}(\Gamma, \Gamma, \Delta, \Delta)$), and hence that $\mathfrak{P}$ is a pullback (2.6.3∗8). Then by (2.6.3∗5) any element of the fiber $\left(\frac{-}{\mathfrak{P}_{\sigma,A}}\right)^{-1}(\mathfrak{P})$ is a pullback. Thus, to prove the lemma it's enough to construct a commuting square

$$\mathfrak{S} \;:\equiv\; \begin{array}{ccc} \Gamma & \xrightarrow{\;\sigma\;} & \Delta \\ {\scriptstyle \overline{a[\sigma]_{\mathrm{t}}}}\big\downarrow & \not\!\!\Rightarrow_{\gamma} & \big\downarrow{\scriptstyle \overline{a}} \\ \Gamma.A[\sigma]_{\mathrm{T}} & \xrightarrow[\sigma^{\cdot A}]{} & \Delta.A \end{array}$$

such that $\frac{\mathfrak{S}}{\mathfrak{P}_{\sigma,A}} = \mathfrak{P}$.

We follow the same basic strategy used in the proofs of the pullback property of type substitution (3.4.1∗2) and comprehension for 2-coherent wild cwfs (3.4.2∗7). Using (3.3∗5), we will define $\gamma :\equiv \mathrm{sub}^{=}(\epsilon_0, \epsilon_1)$ for some

$$\begin{aligned} \epsilon_0 &: \mathrm{p} \diamond \sigma^{\cdot A} \diamond \overline{a[\sigma]_{\mathrm{t}}} = \mathrm{p} \diamond \overline{a} \diamond \sigma, \\ \epsilon_1 &: \mathrm{q}[\sigma^{\cdot A} \diamond \overline{a[\sigma]_{\mathrm{t}}}]_{\mathrm{t}} \downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathrm{T}}{}^{-1} \cdot [^{=}\epsilon_0]_{\mathrm{T}} \cdot [\diamond]_{\mathrm{T}}} = \mathrm{q}[\overline{a} \diamond \sigma]_{\mathrm{t}}, \end{aligned}$$

and then it's enough to show $\frac{\gamma}{\mathrm{p}\beta^{-1}} = \mathfrak{p}$, i.e.

$$\alpha^{-1} \cdot (\mathrm{p}\beta^{-1} * \overline{a[\sigma]_{\mathrm{t}}}) \cdot \alpha \cdot (\mathrm{p} * \mathrm{sub}^{=}(\epsilon_0, \epsilon_1)) \cdot \alpha^{-1} = (\sigma * \mathrm{p}\beta) \cdot \rho \cdot \lambda^{-1} \cdot (\mathrm{p}\beta^{-1} * \sigma) \cdot \alpha.$$

Using (3.3∗8) and (3.3∗11) we solve this equation for ϵ_0 to obtain the candidate definition

$$\epsilon_0 :\equiv \alpha^{-1} \cdot (\mathrm{p}\beta * \overline{a[\sigma]_{\mathrm{t}}}) \cdot \alpha \cdot (\sigma * \mathrm{p}\beta) \cdot \rho \cdot \lambda^{-1} \cdot (\mathrm{p}\beta^{-1} * \sigma) \cdot \alpha.$$

Finally, we have to construct the witness ϵ_1. By calculation, this reduces to the problem of showing that

$$a[\sigma]_{\mathrm{t}}[\mathrm{id}]_{\mathrm{t}} \downarrow^{\mathsf{Tm}}_{e_1} = a[\sigma]_{\mathrm{t}}[\mathrm{id}]_{\mathrm{t}} \downarrow^{\mathsf{Tm}}_{e_2}$$

where

$$e_1 \equiv [^{=}\mathrm{p}\beta^{-1}]_{\mathrm{T}} \cdot [\diamond]_{\mathrm{T}} \cdot ([\diamond]_{\mathrm{T}}{}^{-1} \cdot [^{=}\mathrm{p}\beta^{-1}]_{\mathrm{T}} \cdot [\diamond]_{\mathrm{T}})[\overline{a[\sigma]_{\mathrm{t}}}]_{\mathrm{T}} \cdot [\diamond]_{\mathrm{T}}{}^{-1} \cdot [\diamond]_{\mathrm{T}}{}^{-1} \cdot [^{=}\epsilon_0]_{\mathrm{T}} \cdot [\diamond]_{\mathrm{T}}$$

and

$$e_2 \equiv [\diamond]_{\mathrm{T}}{}^{-1} \cdot [^{=}\rho \cdot \lambda^{-1}]_{\mathrm{T}} \cdot [\diamond]_{\mathrm{T}} \cdot ([^{=}\mathrm{p}\beta^{-1}]_{\mathrm{T}} \cdot [\diamond]_{\mathrm{T}})[\sigma]_{\mathrm{T}} \cdot [\diamond]_{\mathrm{T}}{}^{-1}.$$

But e_1 and e_2 can be seen to be equal using the families (3.1.2∗6) and (3.1.2∗9) of 2-cells. □

3.4.3∗2. The proof we have given for (3.4.3∗1) uses the assumption that the wild cwf $\mathscr{C}$ is 2-coherent. In fact, the square commutes (but is not necessarily a pullback) in *arbitrary* wild cwfs, by calculating directly that both $\overline{a} \diamond \sigma$ and $\sigma^{\cdot A} \diamond \overline{a[\sigma]_{\mathrm{t}}}$ are equal to $(\sigma, a[\sigma]_{\mathrm{t}})$.

3.5. TYPE FORMERS

Up to this point we have only considered the structural theory of wild cwfs. We now briefly turn our attention to type formers. In Part II of this thesis, we will make particular use of Π-types and "partial" universes internal to a wild cwf.

3.5∗1. Definition. Π-*structures.* A Π-*structure* on a wild cwf $\mathscr{C}$ is given by the following components[7] which present the type former, constructor and eliminator of a $\mathscr{C}$-internal Π-type:

$$\begin{aligned} \hat{\Pi} &: (A : \mathsf{Ty}\,\Gamma) \to \mathsf{Ty}\,(\Gamma.A) \to \mathsf{Ty}\,\Gamma \\ \lambda &: \mathsf{Tm}_{\Gamma.A}\,B \to \mathsf{Tm}_\Gamma\,(\hat{\Pi}\,A\,B) \\ \mathsf{app} &: \mathsf{Tm}_\Gamma\,(\hat{\Pi}\,A\,B) \to \mathsf{Tm}_{\Gamma.A}\,B \end{aligned}$$

The constructor λ and eliminator app are required to be mutual inverse equivalences, satisfying the β and η-rules[7]

$$\begin{aligned} \hat{\Pi}\beta &: \mathsf{app}\,(\lambda\,b) = b \quad \text{for all} \quad b : \mathsf{Tm}_{\Gamma.A}\,B \\ \hat{\Pi}\eta &: \lambda\,(\mathsf{app}\,f) = f \quad \text{for all} \quad f : \mathsf{Tm}_\Gamma\,(\hat{\Pi}\,A\,B). \end{aligned}$$

Finally, the type former and constructor respect substitution—the following equations hold:

$$\begin{aligned} \hat{\Pi}[]_\mathsf{T} &: (\hat{\Pi}\,A\,B)[\sigma]_\mathsf{T} = \hat{\Pi}\,(A[\sigma]_\mathsf{T})\,(B[\sigma^{\cdot A}]_\mathsf{T}) \quad \text{and} \\ \lambda[]_\mathsf{t} &: (\lambda\,f)[\sigma]_\mathsf{t} = \lambda\,(f[\sigma^{\cdot A}]_\mathsf{t})\downarrow^{\mathsf{Tm}_\Gamma}_{\hat{\Pi}[]_\mathsf{T}{}^{-1}} \quad \text{where} \quad \sigma : \mathscr{C}(\Gamma, \Delta), \\ &\qquad A : \mathsf{Ty}\,\Delta,\ B : \mathsf{Ty}\,(\Gamma.A), \\ &\qquad f : \mathsf{Tm}_\Gamma\,(\hat{\Pi}\,A\,B). \end{aligned}$$

3.5∗2. If $f : \mathsf{Tm}_\Gamma\,(\hat{\Pi}\,A\,B)$ is a term of a $\mathscr{C}$-internal Π-type, then $\mathsf{app}\,f : \mathsf{Tm}_{\Gamma.A}\,B$ is the "generic application" of f to the variable $x : A$ in the $\mathscr{C}$-context $(\Gamma,\ x : A)$. The application of f to a *specific* term $\Gamma \vdash a : A$ is then given by the term substitution $(\mathsf{app}\,f)[\overline{a}]_\mathsf{t}$ along the section $\overline{a} : \mathscr{C}(\Gamma, \Gamma.A)$ corresponding to a (by (3.2∗4)).

3.5∗3. Examples. *Canonical* Π-*structure on (sub)universe cwfs.* Any universe cwf $\mathscr{U}$ has a Π-structure given as follows:

- If $A : \Gamma \to \mathscr{U}$ and $B : \Sigma\,\Gamma\,A \to \mathscr{U}$ are $\mathscr{U}$-types in contexts Γ and $\Gamma.A$ respectively, then

$$\begin{aligned} \hat{\Pi}\,A\,B &: \Gamma \to \mathscr{U} \\ (\hat{\Pi}\,A\,B)(\gamma) &:\equiv \Pi\,(A\,\gamma)\,(B\,(\gamma, \_)), \end{aligned}$$

- λ and app are, respectively, the currying and uncurrying operations, and
- all equations hold definitionally.

[7]Implicitly quantifying over $\Gamma : \mathscr{C}_0$, $A : \mathsf{Ty}\,\Gamma$ and $B : \mathsf{Ty}\,(\Gamma.A)$ as needed.

Any subuniverse cwf of $\mathcal{U}$—in particular, $\mathrm{Set}_{\mathscr{U}}$—inherits this canonical Π-structure.

3.5∗4. Definition. *Partial universe structures.* A *partial universe structure* on a wild cwf $\mathscr{C}$ consists of components

$$\begin{aligned} \mathbb{U} &: (\Gamma : \mathscr{C}_0) \to \mathsf{Ty}\,\Gamma \\ \mathrm{El} &: (\Gamma : \mathscr{C}_0) \to \mathsf{Tm}\,\mathbb{U}_\Gamma \to \mathsf{Ty}\,\Gamma \end{aligned}$$

and equations

$$\begin{aligned} \mathbb{U}[]_\mathsf{T} &: \mathbb{U}_\Delta[\sigma]_\mathsf{T} = \mathbb{U}_\Gamma && \text{and} \\ \mathrm{El}[] &: (\mathrm{El}\,T)[\sigma]_\mathsf{T} = \mathrm{El}\,(T[\sigma]_\mathsf{t}\downarrow^{\mathsf{Tm}\,\Gamma}_{\mathbb{U}[]_\mathsf{T}}) && \text{where } \Gamma, \Delta : \mathscr{C}_0,\ \sigma : \mathscr{C}(\Gamma, \Delta), \\ & && T : \mathsf{Tm}\,\mathbb{U}_\Gamma. \end{aligned}$$

3.5∗5. Example. *$\mathscr{U}$ as a partial universe in $\mathcal{U}^+$.* Any universe $\mathscr{U}$ gives rise to a partial universe structure on its successor universe cwf $\mathcal{U}^+$, where $\mathbb{U}_\Gamma :\equiv \mathscr{U}$ for all $\Gamma : \mathcal{U}^+{}_0$, and where El_Γ is the usual decoding function for $\mathscr{U}$.

3.5∗6. As its name suggests, a partial universe structure is a partial axiomatization of a universe type internal to a wild cwf. We stress two important points by which partial universes $\mathbb{U}$ in wild cwfs $\mathscr{C}$ are strictly more general than Tarski universes:

1. $\mathbb{U}$ need not be closed under type formers—in particular, not under $\hat{\Pi}$ in the case that $\mathscr{C}$ is equipped with a Π-structure.
2. There is no requirement that $\mathbb{U}$ be an object classifier, i.e. contain codes for "all small types in $\mathscr{C}$".

Thus $\mathbb{U}$ is a $\mathscr{C}$-type whose elements are codes for *some*, but not necessarily all $\mathscr{C}$-types. In particular, $\mathbb{U}$ may contain codes for *no* $\mathscr{C}$-types at all.

3.5∗7. Examples. *The empty and trivial partial universes.* The empty type family

$$\begin{aligned} \mathbb{U}^{\mathbb{0}} &: (\Gamma : \mathscr{U}) \to \Gamma \to \mathscr{U} \\ \mathbb{U}^{\mathbb{0}}_\Gamma(\gamma) &:\equiv \mathbb{0} \end{aligned}$$

gives rise to a partial universe structure on $\mathcal{U}$ with decoding function

$$\begin{aligned} \mathrm{El}^{\mathbb{0}} &: (\Gamma : \mathscr{U}) \to \Pi\,\Gamma\,\mathbb{U}^{\mathbb{0}}_\Gamma \to \Gamma \to \mathscr{U} \\ \mathrm{El}^{\mathbb{0}}_\Gamma(T, \gamma) &:\equiv \mathbb{0}\text{-elim}(T(\gamma)) \end{aligned}$$

Similarly, we have the trivial partial universe

$$\begin{aligned} \mathbb{U}^{\mathbb{1}} &: (\Gamma : \mathscr{U}) \to \Gamma \to \mathscr{U} \\ \mathbb{U}^{\mathbb{1}}_\Gamma(\gamma) &:\equiv \mathbb{1} \end{aligned}$$

on $\mathcal{U}$ that only codes the unit type

$$\begin{aligned} \mathrm{El}^{\mathbb{1}} &: (\Gamma : \mathscr{U}) \to \Pi\,\Gamma\,\mathbb{U}^{\mathbb{1}}_\Gamma \to \Gamma \to \mathscr{U} \\ \mathrm{El}^{\mathbb{1}}_\Gamma(T, \gamma) &:\equiv \mathbb{1}. \end{aligned}$$

These partial universe structures are inherited by any subuniverse cwf (3.1.4∗7).

3.5∗8. Note in particular that the set universe cwf Set does not have a Tarski universe type of sets, but can still be equipped with the empty or trivial partial universe structures.

3.6. TELESCOPES

Throughout this section we assume that $\mathscr{C}$ is a wild cwf.

3.6∗1. *Telescopes.* A *telescope* in context $\Gamma : \mathscr{C}_0$ is a finite dependent extension of Γ by $\mathscr{C}$-types $A_0, A_1, \ldots, A_{n-1}$ for some $n : \mathbb{N}$, where $A_i : \mathsf{Ty}\,(\Gamma.A_0.\ldots.A_{i-1})$ for each $i < n$.

3.6∗2. Definition. *Telescopes and telescope closure.* More precisely, we define the $\mathscr{C}_0$-indexed type family of $\mathscr{C}$-*telescopes*

$$\mathsf{Tel} : \mathscr{C}_0 \to \mathscr{U}$$

by small induction-recursion, simultaneously with the *telescope closure* operation

$$\_ \mathbin{+\!\!+} \_ : (\Gamma : \mathscr{C}_0) \to \mathsf{Tel}\,\Gamma \to \mathscr{C}_0.$$

For every context $\Gamma : \mathscr{C}_0$, the constructors of $\mathsf{Tel}\,\Gamma$ are

$$\begin{aligned} \bullet &: \mathsf{Tel}\,\Gamma \\ \_ \blacktriangleright \_ &: (\Theta : \mathsf{Tel}\,\Gamma) \to \mathsf{Ty}\,(\Gamma \mathbin{+\!\!+} \Theta) \to \mathsf{Tel}\,\Gamma, \end{aligned}$$

and the closure $\Gamma \mathbin{+\!\!+} \Theta$ of a $\mathscr{C}$-telescope $\Theta : \mathsf{Tel}\,\Gamma$ is the context formed by recursively appending Θ to Γ:

$$\begin{aligned} \Gamma \mathbin{+\!\!+} \bullet &:\equiv \Gamma \\ \Gamma \mathbin{+\!\!+} (\Theta \blacktriangleright A) &:\equiv (\Gamma \mathbin{+\!\!+} \Theta).A. \end{aligned}$$

3.6∗3. We also denote the closure of a telescope $\Theta : \mathsf{Tel}\,\Gamma$ by $\overline{\Theta}$ when its context Γ is understood. In this notation, the defining equations of (3.6∗2) are

$$\begin{aligned} \overline{\bullet} &\equiv \Gamma \\ \overline{\Theta \blacktriangleright A} &\equiv \overline{\Theta}.A. \end{aligned}$$

3.6∗4. Definition. *Iterated projection.* For any $\mathscr{C}$-telescope $\Theta : \mathsf{Tel}\,\Gamma$, we have the iterated display map $\mathsf{p}_\Theta : \mathscr{C}(\Gamma \mathbin{+\!\!+} \Theta, \Gamma)$, defined by

$$\begin{aligned} \mathsf{p}_\bullet &:\equiv \mathrm{id}_\Gamma \\ \mathsf{p}_{(\Theta \blacktriangleright A)} &:\equiv \mathsf{p}_\Theta \diamond \mathsf{p}_A, \end{aligned}$$

that forgets the telescope and only projects out its context.

3.6∗5. Thus, any telescope $\Theta : \mathsf{Tel}\,\Gamma$ gives rise to an object $\Gamma \mathbin{+\!\!+} \Theta \xrightarrow{\mathsf{p}_\Theta} \Gamma$ in the (wild) slice over Γ.

3.6∗6. By iterated pasting of type substitution pullbacks (3.4.1∗3) we get the following property of telescopes.

3.6∗7. Lemma. *Substitution through a telescope.* Suppose that $\mathscr{C}$ is 2-coherent and $\Gamma, \Delta : \mathscr{C}_0$. For any telescope $\Theta : \mathsf{Tel}\,\Delta$ and substitution $\sigma : \mathscr{C}(\Gamma, \Delta)$, there is a telescope $\Theta[\sigma] : \mathsf{Tel}\,\Gamma$, a substitution

$$\sigma +\!\!+\, \Theta : \mathscr{C}(\overline{\Theta[\sigma]}, \overline{\Theta})$$

and a 2-cell $\mathsf{t}_{\sigma,\Theta} : \sigma \diamond \mathsf{p}_{\Theta[\sigma]} = \mathsf{p}_{\Theta} \diamond \sigma +\!\!+\, \Theta$ such that the commuting square

$$\mathfrak{T}_{\sigma,\Theta} \quad :\equiv \quad \begin{array}{ccc} \Gamma +\!\!+\, \Theta[\sigma] & \xrightarrow{\sigma +\!\!+ \Theta} & \Delta +\!\!+\, \Theta \\ {\scriptstyle \mathsf{p}_{\Theta[\sigma]}}\big\downarrow & \overset{\mathsf{t}_{\sigma,\Theta}}{\Leftarrow} & \big\downarrow{\scriptstyle \mathsf{p}_{\Theta}} \\ \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array}$$

is a pullback.

Proof. By induction on $\Theta : \mathsf{Tel}\,\Delta$. For the empty telescope, define

$$\begin{aligned} \bullet[\sigma] &:\equiv \bullet \\ \text{and} \quad \sigma +\!\!+ \bullet &:\equiv \sigma, \end{aligned}$$

and then the square

$$\begin{array}{ccc} \Gamma & \xrightarrow{\sigma} & \Delta \\ {\scriptstyle \mathrm{id}}\big\downarrow & \overset{}{\underset{\rho\cdot\lambda^{-1}}{\Leftarrow}} & \big\downarrow{\scriptstyle \mathrm{id}} \\ \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array}$$

is a pullback by (2.6.2∗4).

Inductively, assume $A : \mathsf{Ty}\,\overline{\Theta}$ and define

$$(\Theta \blacktriangleright A)[\sigma] \quad :\equiv \quad \Theta[\sigma] \blacktriangleright A[\sigma +\!\!+\, \Theta]_{\mathsf{T}}.$$

Then

$$\overline{(\Theta \blacktriangleright A)[\sigma]} \quad \equiv \quad \overline{\Theta[\sigma]}.A[\sigma +\!\!+\, \Theta]_{\mathsf{T}}$$

by (3.6∗3), and we define

$$\sigma +\!\!+\, (\Theta \blacktriangleright A) \quad :\equiv \quad (\sigma +\!\!+\, \Theta)^{\cdot A}$$

to be the lift of $\sigma +\!\!+\, \Theta$ given by the type substitution pullback

$$\mathfrak{P}_{\sigma +\!\!+ \Theta, A} \quad \equiv \quad \begin{array}{ccc} \overline{\Theta[\sigma]}.A[\sigma +\!\!+\, \Theta]_{\mathsf{T}} & \xrightarrow{(\sigma +\!\!+ \Theta)^{\cdot A}} & \overline{\Theta}.A \\ \big\downarrow & \overset{}{\underset{\mathsf{p}\beta^{-1}}{\Leftarrow}} & \big\downarrow \\ \overline{\Theta[\sigma]} & \xrightarrow[\sigma +\!\!+ \Theta]{} & \overline{\Theta} \end{array} .$$

The vertical pasting of this pullback with the pullback $\mathfrak{T}_{\sigma,\Theta}$ given by the induction hypothesis has vertical sides $\mathsf{p}_{(\Theta \blacktriangleright A)[\sigma]_{\mathsf{T}}}$ and $\mathsf{p}_{(\Theta \blacktriangleright A)}$, and is a pullback by the pasting lemma (2.6.3∗5). □

3.6∗8. Corollary. If $\mathscr{C}$ is univalent, then for any telescope $\Theta : \mathsf{Tel}\,\Gamma$ and $\mathscr{C}$-type $X : \mathsf{Ty}\,\Gamma$, the pullback squares

$$\mathfrak{T}_{p_X,\Theta} \equiv \begin{array}{ccc} \Gamma.X +\!\!+ \Theta[p_X] & \xrightarrow{p_X +\!+ \Theta} & \Gamma +\!\!+ \Theta \\ {\scriptstyle p_{\Theta[p_X]}}\big\downarrow & \mathfrak{t}_{p_X,\Theta} & \big\downarrow{\scriptstyle p_\Theta} \\ \Gamma.X & \xrightarrow[p_X]{} & \Gamma \end{array}$$

and

$$\mathfrak{P}_{p_\Theta,\, p_X}{}^{\mathsf{T}} \equiv \begin{array}{ccc} (\Gamma +\!\!+ \Theta).X[p_\Theta]_{\mathsf{T}} & \xrightarrow{p_{X[p_\Theta]_{\mathsf{T}}}} & \Gamma +\!\!+ \Theta \\ {\scriptstyle p_\Theta{}^{.X}}\big\downarrow & p\beta & \big\downarrow{\scriptstyle p_\Theta} \\ \Gamma.X & \xrightarrow[p_X]{} & \Gamma \end{array}$$

are equal elements of the proposition $\mathsf{Pullback}(p_X, p_\Theta)$ (2.6.4∗4). It follows that

$$\Gamma.X +\!\!+ \Theta[p_X] = (\Gamma +\!\!+ \Theta).X[p_\Theta]_{\mathsf{T}}$$

by the equality induced by the gap map μ of the type substitution pullback (3.4.1∗3), and also that

$$\begin{aligned} p_{\Theta[p_X]} &= p_\Theta{}^{.X} \diamond \mu, \\ p_X +\!\!+ \Theta &= p_{X[p_\Theta]_{\mathsf{T}}} \diamond \mu. \end{aligned}$$

In words, this is the expected property that weakening a telescope Θ in context Γ by $X : \mathsf{Ty}\,\Gamma$ has the same effect as extending the closure of Θ by X.

3.6∗9. Lemma. *Weakening telescope maps.* Let Θ and Θ' be telescopes in context Γ. If $\sigma : \mathscr{C}(\overline{\Theta}, \overline{\Theta'})$ is a substitution over Γ, i.e. such that $p_{\Theta'} \diamond \sigma = p_\Theta$, then for any $X : \mathsf{Ty}\,\Gamma$ there is a substitution

$$\sigma^{\uparrow X} : \mathscr{C}(\overline{\Theta[p_X]}, \overline{\Theta'[p_X]})$$

over $\Gamma.X$ such that

$$\begin{array}{ccc} \overline{\Theta[p_X]} & \xrightarrow{\sigma^{\uparrow X}} & \overline{\Theta'[p_X]} \\ {\scriptstyle p_X +\!+ \Theta}\big\downarrow & & \big\downarrow{\scriptstyle p_X +\!+ \Theta'} \\ \overline{\Theta} & \xrightarrow[\sigma]{} & \overline{\Theta'} \end{array}$$

can be completed to a pullback.

Proof. By (2.6.3∗12) and (3.6∗7). □

3.6∗10. Definition. *Iterated internal* Π*-types.* If $\mathscr{C}$ is equipped with a Π-structure (3.5∗1), for any context $\Gamma : \mathscr{C}_0$ we define

$$\begin{aligned} \hat{\Pi}^* &: (\Theta : \mathsf{Tel}\,\Gamma) \to \mathsf{Ty}\,(\Gamma +\!\!+ \Theta) \to \mathsf{Ty}\,\Gamma \\ \lambda^* &: (\Theta : \mathsf{Tel}\,\Gamma) \to \mathsf{Tm}_{\Gamma +\!+ \Theta}\,B \to \mathsf{Tm}_\Gamma\,(\hat{\Pi}^*\,\Theta\,B) \\ \mathsf{app}^* &: (\Theta : \mathsf{Tel}\,\Gamma) \to \mathsf{Tm}_\Gamma\,(\hat{\Pi}^*\,\Theta\,B) \to \mathsf{Tm}_{\Gamma +\!+ \Theta}\,B \end{aligned}$$

by

$$\begin{aligned} \hat{\Pi}^* \bullet B &:\equiv B \\ \hat{\Pi}^* (\Theta \blacktriangleright A)\, B &:\equiv \hat{\Pi}^*\, \Theta\, (\hat{\Pi}\, A\, B), \end{aligned}$$

$$\begin{aligned} \lambda^*_\bullet\, b &:\equiv b \\ \lambda^*_{\Theta \blacktriangleright A}\, b &:\equiv \lambda^*_\Theta\, (\lambda\, b), \end{aligned}$$

and

$$\begin{aligned} \mathrm{app}^*_\bullet\, f &:\equiv f \\ \mathrm{app}^*_{\Theta \blacktriangleright A}\, f &:\equiv \mathrm{app}\, (\mathrm{app}^*_\Theta\, f). \end{aligned}$$

3.6∗11. *Substitution in an iterated* Π*-type.* For any telescope $\Theta : \mathsf{Tel}\, \Delta$, $\mathscr{C}$-type $B : \mathsf{Ty}\, \overline{\Theta}$ and substitution $\sigma : \mathscr{C}(\Gamma, \Delta)$, we can show by induction on Θ that

$$(\hat{\Pi}^*\, \Theta\, B)[\sigma]_{\mathsf{T}} = \hat{\Pi}^*\, (\Theta[\sigma])\, (B[\sigma \mathbin{+\!\!+} \Theta]_{\mathsf{T}}).$$

3.6∗12. λ^* *and* app^* *are inverse equivalences.* Furthermore, for any $\Theta : \mathsf{Tel}\, \Gamma$ and $B : \mathsf{Ty}\, \overline{\Theta}$ we have that

$$\mathrm{app}^*_\Theta : \mathsf{Tm}_\Gamma\, (\hat{\Pi}^*\, \Theta\, B) \simeq \mathsf{Tm}_{\Gamma \mathbin{+\!\!+} \Theta}\, B$$

is an equivalence with inverse λ^*_Θ, by induction on Θ and the fact that λ and app are inverse equivalences.

3.6∗13. Definition. *Internal type families.* Suppose that $\mathscr{C}$ is equipped with a Π-structure and a partial universe $\mathbb{U}$ (3.5∗4). For any context $\Gamma : \mathscr{C}_0$ and telescope $\Theta : \mathsf{Tel}\, \Gamma$, the iterated internal Π allows us to define a $\mathscr{C}$-type of Θ-indexed families of $\mathbb{U}$-types,

$$\mathbb{T}_\Theta :\equiv \hat{\Pi}^*\, \Theta\, \mathbb{U}_{\overline{\Theta}} : \mathsf{Ty}\, \Gamma.$$

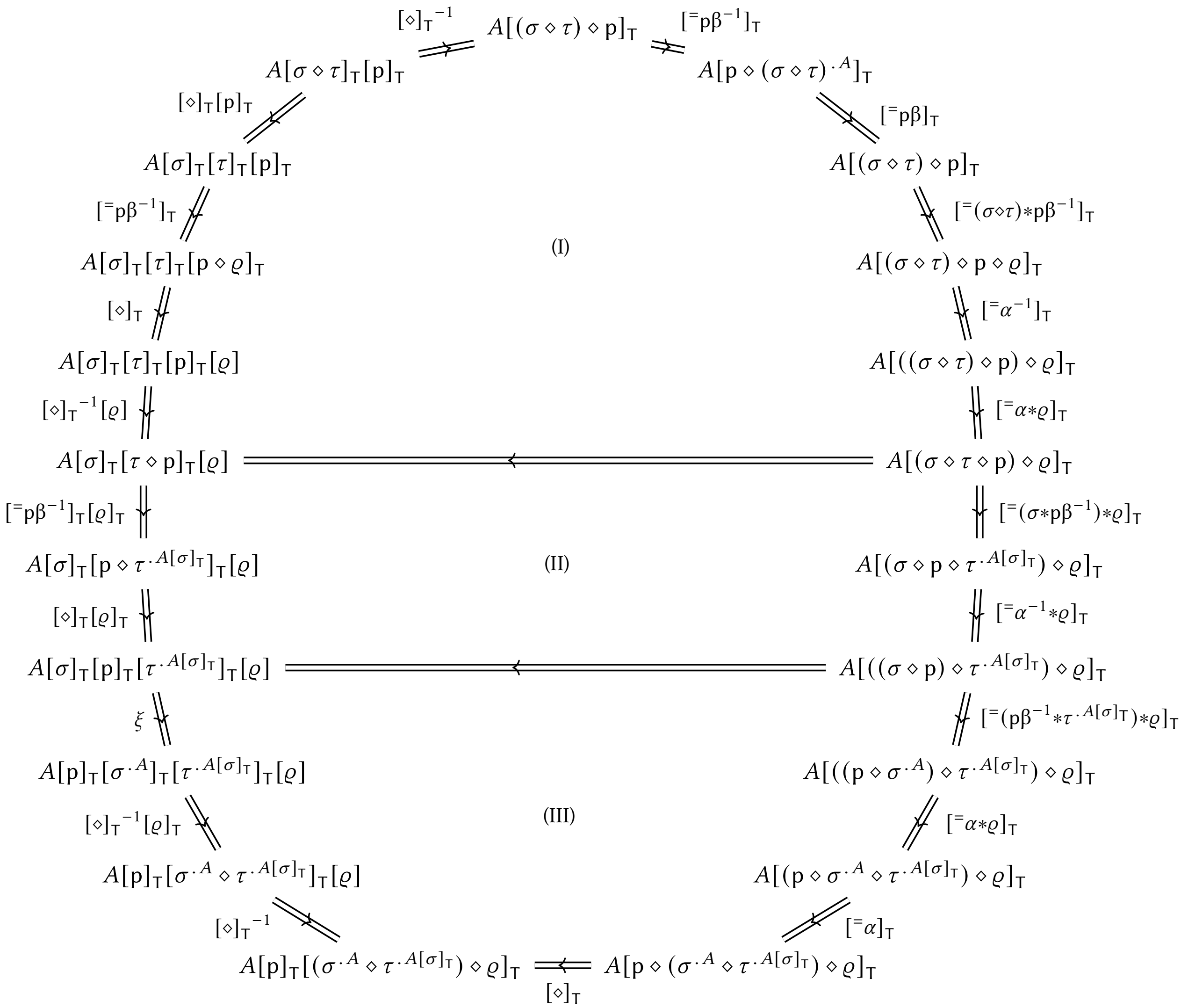
By (3.6∗11), $\mathbb{U}[]_{\mathsf{T}}$ (3.5∗4) and (3.6∗12) we have an equivalence

$$\mathsf{Tm}\, (\mathbb{T}_\Theta[\mathsf{p}_{\mathbb{T}_\Theta}]_{\mathsf{T}}) = \mathsf{Tm}\, (\hat{\Pi}^*\, (\Theta[\mathsf{p}_{\mathbb{T}_\Theta}])\, \mathbb{U}) \simeq \mathsf{Tm}_{\overline{\Theta[\mathsf{p}_{\mathbb{T}_\Theta}]}}\, \mathbb{U}.$$

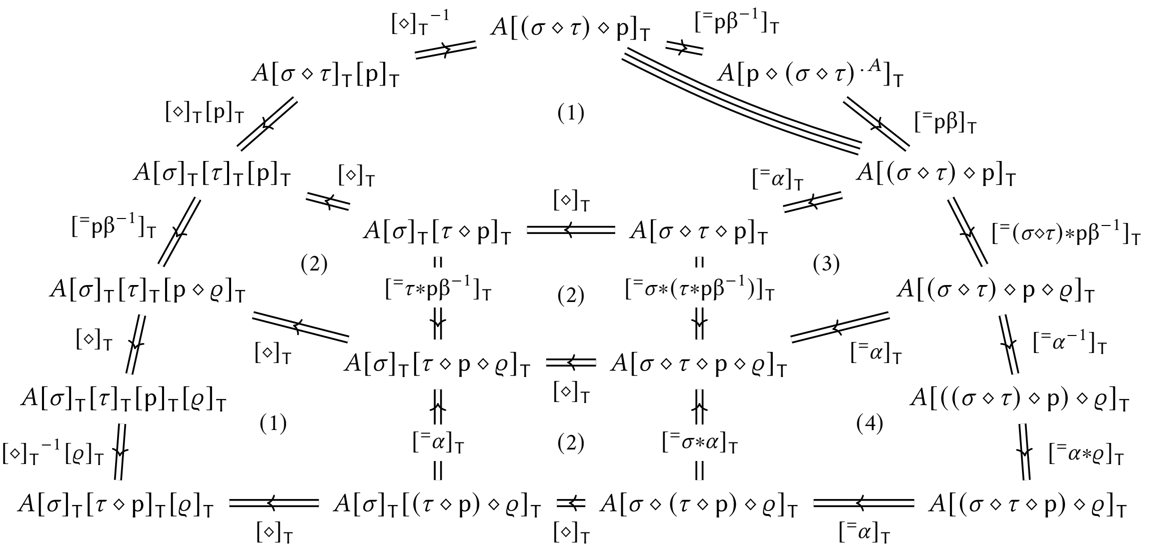
Transporting the generic term $\mathsf{q}_{\mathbb{T}_\Theta}$ along this equivalence and applying El (3.5∗4), we get a $\mathscr{C}$-type

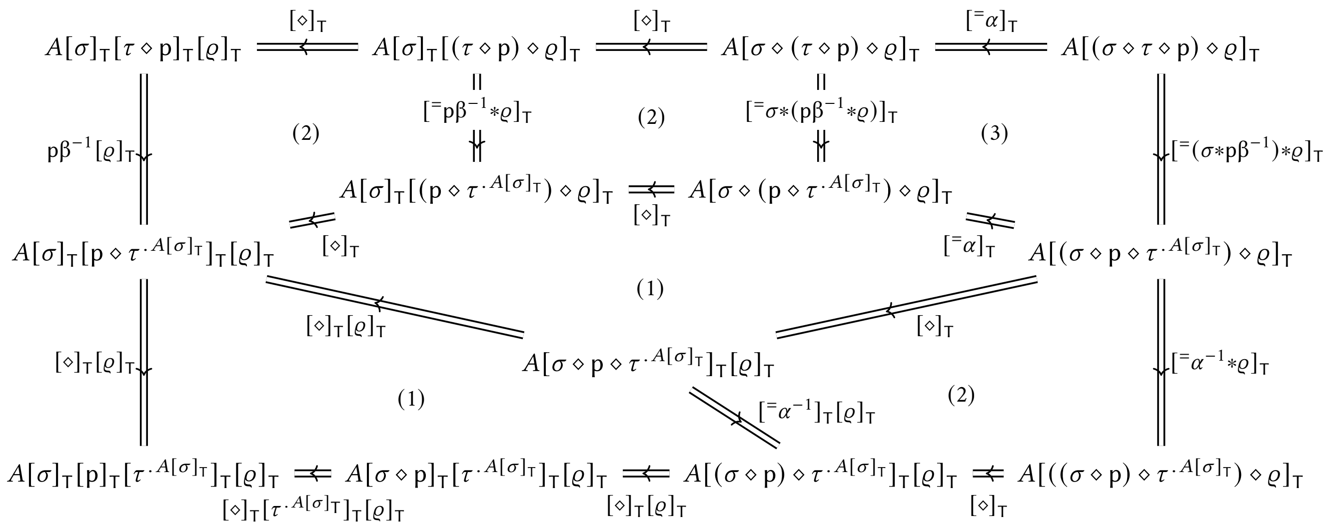
$$\mathsf{Q}_\Theta : \mathsf{Ty}\, (\overline{\Theta[\mathsf{p}_{\mathbb{T}_\Theta}]})$$

which we call the *generic* Θ*-indexed type*.

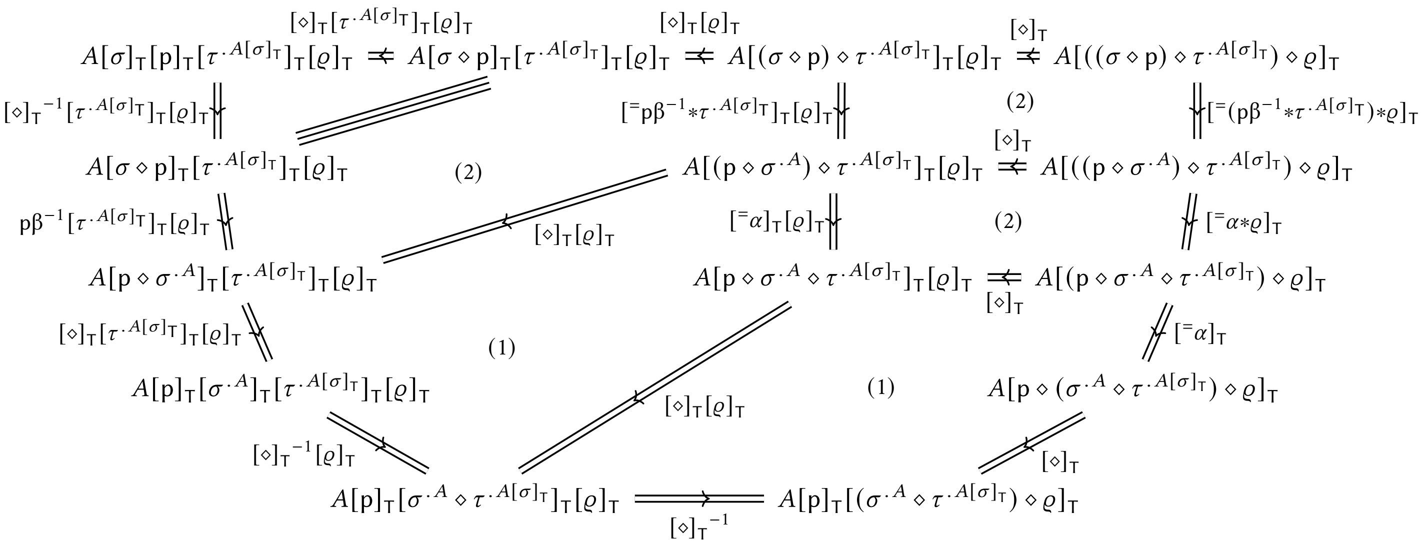


3.4.2∗7∗1. The pasting proof of $e_1 = e_2$ splits into three regions, which are filled with the cells shown in Diagrams 3.4.2∗7∗2 and Diagram 3.4.2∗7∗3. We abbreviate the substitution $(\mathrm{p}, \mathrm{q}_{\downarrow [\diamond]_\mathsf{T}[\mathrm{p}]_\mathsf{T}})$ by ϱ and the substitution $([\diamond]_\mathsf{T}{}^{-1} \cdot [{}^{=}\mathrm{p}\beta^{-1}]_\mathsf{T} \cdot [\diamond]_\mathsf{T})[\tau^{\cdot A[\sigma]_\mathsf{T}}]_\mathsf{T}[\varrho]_\mathsf{T}$ by ξ.


Region (I)


Region (II)

3.4.2∗7∗2. Filling regions (I) and (II) of Diagram 3.4.2∗7∗1 with coherence cells. Regions marked (1) are filled using type pentagonators (3.1.2∗9); those marked (2), by naturality of $[\diamond]_{\mathsf{T}}$ (3.1.2∗6); (3), by associativity of whiskering (2.2∗4); and (4), by the pentagon associator of the category of contexts (2.3∗3).



3.4.2∗7∗3. Filling region (III) of Diagram 3.4.2∗7∗1 with coherence cells. Regions marked (1) are filled using type pentagonators (3.1.2∗9); regions marked (2), by naturality of $[\diamond]_T$ (3.1.2∗6).

Part II

Internal Type Theoretic Inverse Diagrams

Chapter 4

Type-Valued Reedy-Fibrant Inverse Diagrams

In the second part of this thesis we turn our attention to the theory of type-valued Reedy-fibrant inverse diagrams, with the goal of investigating the construction of such diagrams in internal models of HoTT.

We will not make crucial use of the identity morphisms of the index categories in our proposed constructions, and therefore consider diagrams indexed by inverse *semi*categories.

4.1. WELL FOUNDED TYPE-VALUED RELATIONS

4.1∗1. Definition. *Accessibility and well foundedness.* Assume that A is a type with a *type-valued* relation $\_ < \_ : A \to A \to \mathcal{U}$. An element $a : A$ is $<$*-accessible* if, inductively, all its predecessors $a' < a$ are $<$-accessible. The relation $<$ on A is *well founded* if every element of A is $<$-accessible.

4.1∗2. *Well foundedness is a proposition.* By the same proof as for [Uni13, Lemma 10.3.2], $<$-accessibility defines a propositional predicate

$$\text{is-accessible}_< : A \to \text{Prop}_\mathcal{U}.$$

Thus well foundedness of $<$ is also a proposition.

4.1∗3. As in [Uni13, Chapter 10], we have elimination and recursion principles for the accessibility predicate, as well as well founded induction on well founded type-valued relations.

4.1∗4. A type A equipped with a family $\_ < \_ : A \to A \to \mathcal{U}$ is also called a *graph* in the literature. We will continue calling $<$ a relation—always assuming that it is, in general, type-valued.

4.1∗5. Now we consider how to build new well founded relations from existing ones. In particular, we use the notions of *lifted relations* and *subrelations*, as also considered by Paulson in [Pau86].

4.1∗6. Definition. *Lifting relations.* Let $f : A \to B$ be a function, and $<$ a binary relation on B. The *lift* of $<$ along f is the binary relation $<^f$ on A given by

$$a' <^f a \;:\equiv\; f(a') < f(a).$$

4.1∗7. Lemma. *Lifting well founded induction.* Suppose that $f : A \to B$ maps an arbitrary type A to a type B with a well founded relation $\_ < \_ : B \to B \to \mathcal{U}$. Then for any type family $P : A \to \mathcal{U}$, to show that $P(a)$ for all $a : A$ it's enough to show that

$$\prod_{(a:A)} \Big(\prod_{(a':A)} f(a') < f(a) \to P(a')\Big) \to P(a). \tag{$*$}$$

Proof. Well founded induction on $(B, <)$ gives us an eliminator of type

$$\Big(\prod_{(b:B)} \Big(\prod_{(b':B)} b' < b \to Q(b')\Big) \to Q(b)\Big) \to \Pi\, B\, Q$$

for all $Q : B \to \mathcal{U}$. Instantiating Q with the family defined by

$$Q(b) :\equiv \prod_{(a : f^{-1}(b))} P(a)$$

and contracting singletons, we see that the conclusion of the eliminator becomes $\Pi\, A\, P$, and its premise the desired criterion ($*$). □

4.1∗8. Corollary. *Lifting well founded relations.* The lift $<^f$ of a well founded relation $<$ on a type B along any function $f : A \to B$ is well founded.

Proof. We want to show that every $a : A$ is $<^f$-accessible. By (4.1∗7) it's enough to show that, for every $a : A$, if all $a' : A$ satisfying $f(a') < f(a)$ are $<^f$-accessible, then so is a. But this holds by definition of $<^f$ and $<^f$-accessibility. □

4.1∗9. Definition. *Subrelations.* Assume that $<_1$ and $<_2$ are binary relations on a type A. We say that $<_1$ is a *subrelation* of $<_2$ if, for all $a', a : A$,

$$a' <_1 a \to a' <_2 a.$$

4.1∗10. Lemma. *Subrelations of well founded relations.* If $<_1$ is a subrelation of a well founded relation $<_2$ on A, then $<_1$ is well founded.

Proof. We show that every $a : A$ is $<_1$-accessible. By well founded induction along $<_2$ we can assume that a' is $<_1$-accessible for every $a' <_2 a$. Since $<_1$ is a subrelation of $<_2$, this means that a' is $<_1$-accessible for every $a' <_1 a$. □

4.1∗11. The upshot of (4.1∗8) and (4.1∗10) is that if $f : A \to B$ is strictly monotone between relations $(A, <_A)$ and $(B, <_B)$,[1] then $<_A$ is well founded whenever $<_B$ is.

4.1∗12. Definition. *Ordinals.* An *ordinal* is a type α with a *propositional* relation $\_ < \_ : \alpha \to \alpha \to \mathrm{Prop}_{\mathcal{U}}$ that is well founded, transitive, and extensional [Uni13, Definition 10.3.9].

[1] i.e. f is a morphism of graphs.

4.1∗13. *Ordinals are sets.* By a theorem of Escardó [Ec24, `extensionally-ordered-types-are-sets`], every type with an extensional propositional relation is a set. Thus ordinals as defined at (4.1∗12) are sets, and equivalent to ordinals as defined in [Uni13, Definition 10.3.17].

4.2. WILD SEMICATEGORIES AND SEMIFUNCTORS

4.2∗1. Definition. *Wild semicategories.* A *wild semicategory* $\mathscr{C}$ is a model of the generalized algebraic theory of wild categories (2.1∗1), without the requirement that the identity morphisms id and their unitors λ, ρ exist.

We call $\mathscr{C}$ a *semicategory* if each of its hom-types $\mathscr{C}(x, y)$ is a set, and, further, a *set-level semicategory* if $\mathscr{C}_0$ is also a set.

4.2∗2. *On the theory of wild semicategories.* Any wild categorical construction that does not make essential use of identity morphisms transfers directly to wild semicategories. In particular, whiskering is defined in wild semicategories as for wild categories (2.2∗3). We also take the definitions of the opposite of a wild semicategory, and coslice semicategories of a semicategory, for granted.

4.2∗3. *Equivalence and univalence in wild semicategories.* Even though $\mathscr{C}$ is not required to have identity morphisms, we can still define $\mathscr{C}$-equivalences to be neutral morphisms (2.4∗2). Univalence may be defined as *completeness* [CK17, Definition 5.5]; any sufficiently coherent complete wild semicategory is then a wild category [CK17, §5].

4.2∗4. Definition. *Wild semifunctors.* A *wild semifunctor* $F : \mathscr{C} \to \mathscr{D}$ between wild semicategories $\mathscr{C}$ and $\mathscr{D}$ consists of:

- The action on objects: a function $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$.
- The action on morphisms: a dependent function F_1 giving, for every $\mathscr{C}$-morphism $f : \mathscr{C}(x, y)$, a $\mathscr{D}$-morphism

$$F_1(f) : \mathscr{D}(F_0(x), F_0(y)).$$

- Preservation of composition: a family of equations

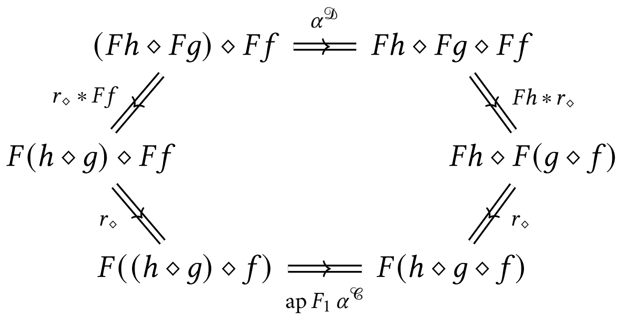
$$r_{\diamond; f,g} : F_1(g) \diamond F_1(f) = F_1(g \diamond f)$$

  indexed over all composable $\mathscr{C}$-morphisms $f : \mathscr{C}(x, y)$ and $g : \mathscr{C}(y, z)$.

As is typical, we usually suppress the subscripts on the object and morphism parts of F, and write Fx and Ff instead of $F_0(x)$ and $F_1(f)$.

4.2∗5. A *semifunctor* is simply a wild semifunctor between semicategories.

4.2∗6. Definition. *2-coherent wild semifunctors.* We call a wild semifunctor F *2-coherent* if it additionally respects associators, in the sense that it is equipped with a family of coherences $r_{\alpha;f,g,h}$ that witness the commutativity of the diagram of equalities of $\mathscr{D}$-morphisms



for each composable triple of $\mathscr{C}$-morphisms f, g and h.

4.2∗7. In this thesis we will consider diagrams in the form of wild semifunctors *into* wild categories, but *indexed over* bona fide semicategories, which will even be *set-level*.

4.2∗8. We will also make crucial use of the following notion, which is widely attributed to Makkai [Mak95].[2]

4.2∗9. Definition. *Finite fan-out.* We say that a semicategory $\mathscr{C}$ has *finite fan-out* if its coslices are finite—for every $x : \mathscr{C}_0$ we have $n : \mathbb{N}$ and a finite enumeration $(x/\mathscr{C})_0 \simeq \mathrm{Fin}(n)$.

4.3. COUNTABLE INVERSE SEMICATEGORIES

4.3∗1. Definition. *Inverse semicategories.* A semicategory $\mathscr{I}$ is *inverse* if the *precedence* relation

$$\_ \prec \_ : \mathscr{I}_0 \to \mathscr{I}_0 \to \mathscr{U}$$
$$j \prec i \;:\equiv\; \mathscr{I}(i, j)$$

on $\mathscr{I}_0$ is well founded.

4.3∗2. Example. *Ordinal inverse semicategories.* Any ordinal $(\alpha, <)$ defines a set-level semicategory with objects $\alpha_0 :\equiv \alpha$ and hom-sets (in fact, *propositions*) $\alpha(x, y) :\equiv x < y$. Since $<$ is transitive and propositional, α has associative composition, and the opposite semicategory α^{op} is inverse.

4.3∗3. *Degree maps.* A semifunctor from an arbitrary semicategory $\mathscr{C}$ to an ordinal inverse semicategory α^{op} is given by a function $\deg : \mathscr{C}_0 \to \alpha$ (the action on objects) that satisfies

$$x \prec y \to \deg(x) < \deg(y)$$

for all $x, y : \mathscr{C}_0$ (the action on morphisms). In this case, the relation $\prec$ is a subrelation of $<^{\deg}$, and by (4.1∗8) and (4.1∗10), well founded. Thus $\mathscr{C}$ is in fact inverse. We call any such semifunctor deg a *degree map* of $\mathscr{C}$.

[2] Who considered them in his development of a semantics for FOLDS (*First Order Logic with Dependent Sorts*), an equivalence-respecting first-order language for mathematical structures.

4.3∗4. In particular, the degree $\deg(i)$ of an object $i : \mathcal{I}_0$ of an inverse semicategory is an upper bound on the length of sequences of morphisms out of i.

4.3∗5. To simplify our discussion of Reedy-fibrant inverse diagrams in Section 4.4 and avoid a lengthy detour into the constructive theory of ordinals, we will work with inverse semicategories whose objects can be identified with (initial segments of) $\mathbb{N}$.

4.3∗6. Definition. *Countable inverse semicategories.* An inverse semicategory $\mathcal{I}$ is called *countable* if it has a degree map $\mathcal{I} \longrightarrow \omega^{\mathrm{op}}$ whose action on objects is an equivalence of types. In particular, there is a countable enumeration $\deg : \mathcal{I}_0 \simeq \mathbb{N}$ of the objects of $\mathcal{I}$.

4.3∗7. *Objects of countable inverse semicategories.* Thus, countable inverse semicategories $\mathcal{I}$ are set-level, and $\mathcal{I}_0$ is a total order.

4.3∗8. *Strict countable inverse semicategories.* We call a countable inverse semicategory $\mathcal{I}_0$ *strict* if it has the identity isomorphism as a degree map. In this case, $\mathcal{I}_0 \equiv \mathbb{N}$. Note that being a strict countable inverse semicategory is a metatheoretic notion.

4.3∗9. Every countable inverse semicategory $\mathcal{I}$ is equivalent to a strict one, and we therefore denote the total order on $\mathcal{I}_0$ by $<$, and refer to objects $i : \mathcal{I}_0$ by their degrees $i : \mathbb{N}$. In particular, $0 : \mathcal{I}_0$, and $i + 1 : \mathcal{I}_0$ whenever $i : \mathcal{I}_0$.

4.3∗10. Definition. *Restrictions of countable inverse semicategories.* For any countable inverse semicategory $\mathcal{I}$ and $i : \mathcal{I}_0$, we denote the full subsemicategories of $\mathcal{I}$ on objects $j < i$ and $j \leq i$ by $\mathcal{I}_{<i}$ and $\mathcal{I}_{\leq i}$ respectively.

4.4. STRICT REEDY-FIBRANT INVERSE DIAGRAMS

In this section we recall, following Shulman [Shu15] and Annenkov et al. [ACKS23], how countable inverse diagrams into categories of types with "enough" limits might be defined *inductively*. In the type theory literature, such diagrams are called *(strictly) Reedy-fibrant*. They simultaneously specialize Reedy-fibrant functors from Reedy categories $\mathcal{C}$ to model categories $\mathcal{M}$ [RV14, Theorem 4.18] to require $\mathcal{C}$ to be inverse, and generalize to allow $\mathcal{M}$ to be a type theoretic fibration category [Shu17, Definition 7.1].

4.4∗1. Warning. *Foundational setting of this section.* This is the one section of this thesis that *cannot* be read in plain homotopical type theoretic foundations, as the usual definitions of type-valued Reedy-fibrant inverse diagrams and their matching objects make use of *definitional* or *strict* equality.

This section may instead be read in a standard categorical setting, taking diagrams into type theoretic fibration categories; or alternatively in two-level type theory (2LTT) [ACKS23], considering diagrams into universes of fibrant types. We adopt this latter setting as it allows us to more directly set up the construction that we later attempt to carry over to homotopical type theory.[3]

4.4∗2. We thus assume familiarity with the concepts of 2LTT, in particular with the distinction between *inner*, *outer* and *fibrant* types [ACKS23, §§2.1–2.3, 3.2]. In contrast to the notation of [ACKS23], we denote the outer naturals by $\mathbb{N}^s$, and continue to write $\mathcal{U}$ for the *inner* universes. We let Type denote a universe of *fibrant* types, which in particular is closed under strong Σ-types.[4] Finally, we implicitly convert from inner to outer types when needed, and elide explicitly writing the conversion operator.

4.4∗3. Warning. *Outer-level index semicategories.* All the definitions of this chapter up to this point also hold on the outer level of a 2LTT, since (with the exception of this section) we work in a neutral setting. In particular, following Section 4.3 we can define *outer* types of inverse and countable inverse semicategories, with all their associated structure. Crucially, for any outer countable inverse semicategory $\mathcal{I}$ we have $\mathcal{I}_0 \simeq \mathbb{N}^s$ via the degree map. In this section we assume that all countable inverse semicategories $\mathcal{I}$ are outer.

4.4∗4. Desideratum. *Strict diagrams over countable inverse semicategories.* Let $\mathcal{I}$ be a countable inverse semicategory. We want to construct a type whose elements are $\mathcal{I}$-indexed diagrams X into Type. Such diagrams consist of

- an object X_i : Type for every $i : \mathcal{I}_0$, and
- a morphism $X_f : X_i \to X_j$ for every $\mathcal{I}$-morphism $f : \mathcal{I}(i, j)$

such that any equality relation on $\mathcal{I}$-morphisms is preserved by the action of X. Subject to the condition, which we will shortly explicate, that Type has certain limits, we will be able to ask that these relations on morphisms be *strictly* preserved. That is, we seek to construct a type of *strict* Type-valued diagrams, i.e. outer-level semifunctors $X : \mathcal{I} \to$ Type such that preservation of composition (4.2∗4) holds for the *outer* equality type. A key idea is the following construction.

4.4∗5. *Extending restrictions of strict countable inverse diagrams.* Given an object $i : \mathcal{I}_0$ of a countable inverse semicategory $\mathcal{I}$, any extension of a strict diagram $X : \mathcal{I}_{<i} \to$ Type to a strict diagram $X' : \mathcal{I}_{\leq i} \to$ Type is given by the data of an object X_i : Type together with appropriate morphisms out of X_i. The universal such extension is the *matching object* $\mathrm{M}_i X$ at i.

4.4∗6. Definition. *Matching objects.* For any countable inverse semicategory $\mathcal{I}$, object $i : \mathcal{I}_0$ and strict diagram $X : \mathcal{I}_{<i} \to$ Type, the *matching object* $\mathrm{M}_i X$ at i is defined to be the strict fibrant limit

$$\mathrm{M}_i X :\equiv \lim_{i/\mathcal{I}} (X \circ \mathrm{cod}).$$

[3] In [ACKS23, §4], Annenkov et al. actually consider diagrams into universes of *outer* types; we do not do this in light of our attempted generalization.

[4] Or an arbitrary type theoretic fibration category if working à la Shulman, in which case Σ is, as usual, the left adjoint to pullback along fibrations.

Explicitly, the terminal cone $\mathrm{M}_i X$ is to consist of an object M_i : Type together with morphisms $M_f : M_i \to X_j$ for every $f : \mathcal{J}(i, j)$, such that for all commuting triangles in $\mathcal{J}$

$$\begin{array}{ccc} & i & \\ {}^{f}\swarrow & & \searrow^{f'} \\ j & \xrightarrow[g]{} & j' \end{array},$$

the triangle in Type

$$\begin{array}{ccc} & M_i & \\ {}^{M_f}\swarrow & & \searrow^{M_{f'}} \\ X_j & \xrightarrow[X_g]{} & X_{j'} \end{array}$$

commutes strictly, i.e. up to outer equality.

4.4∗7. *Extending restrictions of strict countable inverse diagrams, continued.* If the matching object $\mathrm{M}_i X$ can be constructed for the strict diagram X of (4.4∗5), we can then extend X to X' by giving a family of fibrant types $A_i : M_i \to \mathrm{Type}$, and defining

$$X_i :\equiv \textstyle\sum_{(m \,:\, M_i)} A_i(m)$$

as the total space of A_i. The functions X_f for $f : \mathcal{J}(i, j)$ are then given by

$$X_f :\equiv M_f \circ \mathrm{fst},$$

and thus the relations on morphisms in $\mathcal{J}_{\le i}$ are also strictly preserved by X'. Assuming univalence for inner types, the straightening-unstraightening theorem [Uni13, Theorem 4.8.3] implies that *any* extension of X along $\mathcal{J}_{<i} \hookrightarrow \mathcal{J}_{\le i}$ is equivalent to one defined in this way.

4.4∗8. *Fibrant classifiers of restricted countable inverse diagrams.* For any countable inverse $\mathcal{J}$ and $i : \mathcal{J}_0$, the type of strict diagrams $X_{\le i} : \mathcal{J}_{\le i} \to \mathrm{Type}$ is generally not fibrant. However, if "enough" matching objects are constructible, then for any $i : \mathcal{J}_0$ there is a large[5] fibrant type $\mathbb{R}_i$ that *classifies* such strict diagrams up to pointwise equivalence [ACKS23, §§4.4, 4.5]. That is, elements $R : \mathbb{R}_i$ yield strict diagrams $\overline{R} : \mathcal{J}_{\le i} \to \mathrm{Type}$, and every strict diagram $X_{\le i} : \mathcal{J}_{\le i} \to \mathrm{Type}$ is pointwise equivalent to $\overline{R}$ for some $R : \mathbb{R}_i$. Such an $\overline{R}$ is called a *(strict) Reedy-fibrant replacement* of $X_{\le i}$, and $\mathbb{R}_i$ itself is called a *classifier of (strict) i-restricted Reedy-fibrant $\mathcal{J}$-diagrams.*

4.4∗9. Construction. *Inductive construction of fibrant classifiers of restricted Reedy-fibrant diagrams.* Assuming that matching objects are constructible, (4.4∗7) allows us to give an explicit description of the classifier $\mathbb{R}_i : \mathrm{Type}^+$ for arbitrary $i : \mathcal{J}_0$ by induction on the total order $<$ on $\mathcal{J}_0$:

- In the base case, the object $0 : \mathcal{J}_0$ is $\prec$-minimal: there are no morphisms out of 0. A strict diagram $X_{\le 0} : \mathcal{J}_{\le 0} \to \mathrm{Type}$ is then simply given by a fibrant type X_0 : Type, or equivalently, an *inner* type $X_0 : \mathcal{U}$. We define

$$\mathbb{R}_0 :\equiv \mathcal{U},$$

[5] That is, in Type^+.

classifying the constant diagrams. Then $\mathbb{R}_0$ is fibrant.

- Inductively for $i : \mathcal{I}_0$, we define

$$\mathbb{R}_{i+1} :\equiv \sum_{(R : \mathbb{R}_i)} (\mathrm{M}_{i+1}\overline{R} \to \mathcal{U}),$$

  where we use the assumption that fibrant matching objects $\mathrm{M}_{i+1}\overline{R}$ are constructible for all $R : \mathbb{R}_i$. Then $\mathbb{R}_{i+1}$ is fibrant by induction, and any $(R, A_{i+1}) : \mathbb{R}_{i+1}$ yields a strict diagram $\mathcal{I}_{\leq(i+1)} \to \mathrm{Type}$ by extending $\overline{R}$ with the total space of A_{i+1}, as described at (4.4∗7).

4.4∗10. To ensure that matching objects of restricted diagrams can be constructed—so that the inductive construction (4.4∗9) can actually be completed—we further restrict their index semicategories to those with finite fan-out (4.2∗9) (formulated using the outer level). By results of Shulman [Shu15, §11] and Annenkov et al. [ACKS23, §4] we then have the following.

4.4∗11. Theorem. *Constructibility of matching objects.* If a countable inverse semicategory $\mathcal{I}$ has finite fan-out, then for all $i : \mathcal{I}_0$ and restricted strict diagrams $X : \mathcal{I}_{<i} \to \mathrm{Type}$, the (fibrant) matching object $\mathrm{M}_i X$ is constructible using the outer level. This follows by [ACKS23, Theorem 4.8] and induction on $i : \mathcal{I}_0$.

4.4∗12. Definition. *Restricted Reedy-fibrant $\mathcal{I}$-types.* Assume that $\mathcal{I}$ is a countable inverse semicategory with finite fan-out. For $i : \mathcal{I}_0$, elements of the classifier $\mathbb{R}_i : \mathrm{Type}^+$ defined at (4.4∗9) are called *i-restricted (Reedy-fibrant) $\mathcal{I}$-types.*[6] We also call $\mathbb{R}_i$ the *type of i-restricted $\mathcal{I}$-types.*

4.4∗13. *Indexed presentation of restricted Reedy-fibrant diagrams.* Explicitly, we see that an i-restricted $\mathcal{I}$-type $R \equiv (A_0, A_1, \ldots, A_i)$ is a tuple of type families

$$\begin{aligned}
&A_0 : \mathcal{U}\\
&A_1 : \mathrm{M}_1\overline{A_0} \to \mathcal{U}\\
&A_2 : \mathrm{M}_2\overline{(A_0, A_1)} \to \mathcal{U}\\
&\quad\vdots\\
&A_i : \mathrm{M}_i\overline{(A_0, \ldots, A_{i-1})} \to \mathcal{U},
\end{aligned}$$

which gives the "indexed presentation" of a type-valued diagram indexed by $\mathcal{I}_{\leq i}$.

4.4∗14. Now we return to the desideratum (4.4∗4) and ask: Given a countable inverse $\mathcal{I}$, under what assumptions can we construct a type of strict $\mathcal{I}$-indexed type-valued diagrams? We use the following observation, which simply restates the construction (4.4∗9) in light of (4.4∗11).

4.4∗15. Corollary. If $\mathcal{I}$ is a countable inverse semicategory with finite fan-out, then by (4.4∗11) the construction (4.4∗9) yields an outer level function

$$\mathbb{R} : \mathcal{I}_0 \to \mathrm{Type}^+,$$

[6] Following the mathematical tradition of calling $\mathcal{C}$-indexed presheaves "$\mathcal{C}$-sets".

whose values are the types of restricted $\mathcal{J}$-types.

4.4∗16. *Classifiers of countable inverse diagrams.* We may thus give a first answer to (4.4∗4): strict countable inverse diagrams $\mathcal{J} \to \text{Type}$ are classified by the sequential limit L [ACS15, Definition 9] of the outer level family

$$\mathbb{N}^s \simeq \mathcal{J}_0 \xrightarrow{\mathbb{R}} \text{Type}^+ \to \mathcal{U}^+, \tag{$*$}$$

where the map $\text{Type}^+ \to \mathcal{U}^+$ sends a fibrant type to its associated inner type. This is, unfortunately, unsatisfactory, as the outer type L is not known to be fibrant, and is therefore not amenable to further homotopy type theoretic constructions.

What is sought instead is a solution to a longstanding open problem in plain HoTT—the type theoretic construction of a *fibrant* classifier of countable inverse diagrams. One partial answer is the one given by Annenkov et al.: in models of 2LTT in which exponentiating by $\mathbb{N}^s$ preserves fibrant types, such as those in which $\mathbb{N}^s$ is *cofibrant* [ACKS23, Definition 3.13, Remark 3.14], the family ($*$) is fibrant, and its sequential limit is then a *fibrant* classifier of *unrestricted* Reedy $\mathcal{J}$-types.

4.4∗17. Example. *Semisimplicial types.* The subsemicategory spanned by the coface maps of the opposite simplex category $\mathbb{\Delta}^{\text{op}}$ is a countable inverse semicategory ${\mathbb{\Delta}_+}^{\text{op}}$ with finite fan-out. Unrestricted Reedy-fibrant ${\mathbb{\Delta}_+}^{\text{op}}$-types are called *semisimplicial types*, and it is an old open question if the type of unrestricted semisimplicial types can be constructed entirely within plain HoTT.

4.4∗18. *On the provenance of Reedy-fibrant diagrams.* The inductive definition of Reedy-fibrant diagrams via their matching objects originates in categorical homotopy theory; specifically, from the theory of the Reedy model structure on diagrams into model categories [Rie14, Ch. 14, Lemma 14.2.10; RV14, Lemma 3.10]. Their homotopy type theoretic incarnation is due to Shulman [Shu15], who specialized to inverse index categories while weakening the target categories to admit simple type theoretic fibration categories, and developed their theory. The particular formulation of classifiers of Reedy-fibrant type-valued diagrams in 2LTT, and their application to the construction of semisimplicial types, is by Annenkov et al. [ACKS23].

Chapter 5

Indexing Type Theoretic Inverse Diagrams

We aim to study the possibility of developing the construction (4.4 ∗ 9) of classifiers of Reedy-fibrant types in the setting of plain HoTT.[1] This is not simply a matter of reproducing the entirety of Section 4.4 in plain HoTT, as the theory therein makes heavy use of the strict, non-homotopical outer equality of 2LTT.

In particular, in the 2LTT approach one first constructs the matching objects of countable inverse diagrams as outer limits, before proving them fibrant. Of course, in plain HoTT we do not have this technique at our disposal, and instead have to directly construct matching objects entirely homotopically.

To this end, we further restrict the diagrams we consider to those indexed by the *countably simple* semicategories. These are a class of countable inverse semicategories with finite fan-out, which will be more amenable to an inductive specification of matching objects.

This chapter contains joint work with Nicolai Kraus.

5.1. COUNTABLY SIMPLE SEMICATEGORIES

5.1 ∗ 1. Definition. *Countably simple semicategories.* We define a *countably simple semicategory* to be a countable inverse semicategory whose hom-sets are finite. That is, for all $i, j : \mathcal{I}_0$ there is an $n : \mathbb{N}$ and an enumeration

$$[\_]^i_j : \mathrm{Fin}(n) \xrightarrow{\sim} \mathcal{I}(i, j).$$

We denote $[\_]^i_j$ by $[\_]$ when i and j are inferrable from context, and write idx for its inverse isomorphism.

[1] Where, as we later explain, plain HoTT takes the place of the outer level of 2LTT, and a 2-coherent wild cwf (3.3 ∗ 13), the place of the inner level.

5.1∗2. *Hom-sets of countably simple semicategories.* Thus, the hom-sets of countably simple semicategories have decidable equality. The finite enumerations of each hom-set induce decidable total orders which we also denote $<$, so that for each $i, j : \mathcal{I}_0$ we have $[s]^i_j < [t]^i_j$ exactly when $s < t$.

5.1∗3. *Countably simple semicategories have finite fan-out.* For each $i : \mathcal{I}_0$ of a countably simple $\mathcal{I}$, the semicategories $\mathcal{I}_{<i}$, $\mathcal{I}_{\leq i}$, and, crucially, the coslice $i/\mathcal{I}$ are all finite. In particular, countably simple semicategories have finite fan-out (4.2∗9).

5.1∗4. *On definitions of "simple" semicategories.* In the literature, a simple inverse (semi-) category [Mak95; RV22, Definition 11.2.1] is usually taken to simply be an inverse (semi-) category with finite fan-out. In our constructive setting, being inverse and having finite fan-out does not guarantee finite, but only *subfinite* hom-sets. We include the stronger requirement of explicitly ordered hom-sets in our definition; we will later need the order to define linear cosieves, which we use to inductively specify matching objects.

5.1∗5. In order to greatly reduce the complexity of the data we have to keep track of in later constructions, we impose one final constraint on countably simple semicategories $\mathcal{I}$. However, in principle we do not need this restriction; the construction of $\mathcal{I}$-diagrams should generalize equally well to cases where it does not hold.

5.1∗6. Definition. *Strictly oriented countably simple semicategories.* A countably simple semicategory $\mathcal{I}$ is *strictly oriented* if, for any $f : \mathcal{I}(i, j)$ and $k : \mathcal{I}_0$, the precomposition map

$$\_ \diamond f : \mathcal{I}(j, k) \to \mathcal{I}(i, k)$$

is strictly monotone with respect to $<$.

5.1∗7. Non-example. Not every countably simple semicategory is equivalent to a strictly oriented one—a simple counterexample is given by considering the semicategory generated by

$$0 \underset{g}{\overset{f}{\leftleftarrows}} 1 \underset{b}{\overset{a}{\leftleftarrows}} 2$$

with relations $f \diamond a = g \diamond b$ and $f \diamond b = g \diamond a$, but such that $f \diamond a \neq f \diamond b$.

5.1∗8. Examples. However, some important basic semicategories of geometric shapes, such as the ones spanned by the coface maps of the opposite simplex category ($\mathbb{\Delta}_+$) and opposite cube category ($\square_+$), *are* strictly oriented. For $\mathbb{\Delta}_+$ this is by the lexicographic order on hom-sets via its description as the category of finite ordinals and strict monotone maps, and similarly for $\square_+$ via its description as the category of powers of $\{0, 1\}$ and the appropriate face maps.

5.1∗9. Lemma. By decidability of equality and the order $<$ on hom-sets, every morphism of a strictly oriented countably simple semicategory is epi.

5.2. COSIEVES OF COUNTABLY SIMPLE SEMICATEGORIES

For the rest of the chapter, let $\mathcal{I}$ denote an arbitrary countably simple semicategory.

Recall that the matching objects of a Reedy-fibrant $\mathcal{I}$-diagram are limits of a functor over coslices of $\mathcal{I}$. In order to inductively define matching objects, we consider a refinement of the coslices, given by particular (local) cosieves in $\mathcal{I}$.

5.2.1. Cosieves in countably simple semicategories

5.2.1∗1. Definition. *Decidable subsets.* A *decidable subset* of a set A is specified by a characteristic function of type $A \to \mathbb{2}$. We take the basic theory of decidable subsets for granted, and use set theoretic notation as usual.

5.2.1∗2. Definition. *Decidable cosieves.* A *decidable cosieve S under an object i in $\mathcal{I}$* is a decidable subset of $(i/\mathcal{I})_0$ that is closed under postcomposition by arbitrary $\mathcal{I}$-morphisms. We call i the *apex* of S.

5.2.1∗3. For brevity we simply call decidable cosieves, *cosieves*. Since the coslices $(i/\mathcal{I})_0$ are finite, it is decidable if a given cosieve is empty.

5.2.1∗4. Definition. *Cosieve restriction.* Suppose that S is a cosieve under i in $\mathcal{I}$. For any $f : \mathcal{I}(i, j)$, the *restriction of S along f* is the cosieve

$$S \cdot f :\equiv \{g : \mathcal{I}(j, k) \mid k : \mathcal{I}_0,\ g \diamond f \in S\}.$$

5.2.1∗5. It is straightforward to verify that $S \cdot f$ is a cosieve under j for any cosieve S under i and morphism $f : \mathcal{I}(i, j)$. By expanding out definitions one also shows:

5.2.1∗6. Lemma. *Restriction along composites.* Assume that S is a cosieve under i, $f : \mathcal{I}(i, j)$ and $g : \mathcal{I}(j, k)$. Then

$$S \cdot (g \diamond f) = (S \cdot f) \cdot g.$$

5.2.1∗7. We also have the important notion of the height of a cosieve.

5.2.1∗8. Definition. *Cosieve height.* The *height* $\mathfrak{h}(S)$ of a nonempty cosieve S under i in $\mathcal{I}$ is the largest $h : \mathcal{I}_0$ such that $S \cap \mathcal{I}(i, h)$ is nonempty. If S is empty, its height is defined to be $0 : \mathcal{I}_0$.

5.2.1∗9. Definition. *Full cosieves.* A cosieve S under i in $\mathcal{I}$ is called *full* if it is equal to $(i/\mathcal{I})_0$.

5.2.2. Linear cosieves

Now, we focus our attention on a particular class of cosieves that have more structure to exploit.

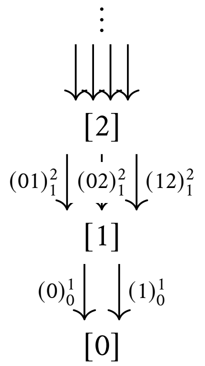
5.2.2∗1. Definition. *Linear cosieves.* A cosieve S under i of height h is *linear* if

1. $\mathcal{I}(i, j) \subseteq S$ for all objects $j < h$, and

2. $S \cap \mathcal{I}(i, h)$ is a $<$-prefix of $\mathcal{I}(i, h)$.

In particular, the empty cosieve with apex i is linear.

5.2.2∗2. Example. *Linear and non-linear cosieves.* Consider the following initial segment of (the semicategory part of) Δ_+^{op},

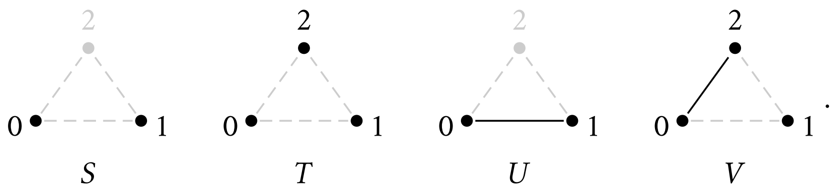


where composites of coface maps are presented by listing the vertices of the corresponding face, and hom-sets are ordered lexicographically with respect to this presentation. Now consider the following cosieves under $[2]$:

$$\begin{aligned} S &:\equiv \{(0)^2_0, (1)^2_0\}, \\ T &:\equiv \{(0)^2_0, (1)^2_0, (2)^2_0\}, \\ U &:\equiv \{(0)^2_0, (1)^2_0, (01)^2_1\}, \\ V &:\equiv \{(0)^2_0, (1)^2_0, (2)^2_0, (02)^2_1\}. \end{aligned}$$

It's easy to check that S and T are linear. U is not linear since it has height $[1]$ but we do not have $\Delta_+^{op}([2], [0]) \subseteq U$, i.e. U is "missing the 0-cell $(2)^2_0$". V is also not linear since $V \cap \Delta_+^{op}([2], [1]) = \{(02)^2_1\}$ is not a $<$-prefix of $\Delta_+^{op}([2], [1])$, i.e. V "skips over the 1-cell $(01)^2_1$".

5.2.2∗3. We intuit a cosieve $W \subseteq (i/\mathcal{I})_0$ as indexing a "subskeleton" of the "boundary shape $\partial\bigcirc^i$ of the i-cell $\bigcirc^i$ indexed by $(i/\mathcal{I})_0$". The "dimension" d of this subskeleton is given by the height $\mathfrak{h}(S)$ of W. According to this intuition, the cosieves in (5.2.2∗2) index the following subskeleta of the 2-simplex boundary $\partial\Delta^2$,



Then the conditions of (5.2.2∗1) say that for a cosieve to be linear it must

1. for every $k < d$, "index all $(k-1)$-faces of $\partial\bigcirc^i$ before it indexes any k-face", and
2. for every "$\mathfrak{h}(S)$-dimensional face" f indexed by S, S must also "index every $\mathfrak{h}(S)$-face that comes before f".

5.2.2∗4. Lemma. *Linear cosieves under a fixed apex form ordered sets.* Linear cosieves under the same apex are linearly ordered by $\subseteq$. This is because any two distinct cosieves S, T under an object i are comparable: if they have the same height h then the one with the longer $<$-prefix of $\mathcal{I}(i,\, h)$ contains the other, otherwise the one with the greater height contains the other.

5.2.2∗5. Lemma. *Restriction of linear cosieves.* When $\mathcal{I}$ is *strictly oriented* (5.1∗6), the restriction of a linear cosieve S under i in $\mathcal{I}$ along any morphism $f : \mathcal{I}(i,\, j)$ is again a linear cosieve, under j.

Proof. If S is empty then $S \cdot f$ is empty, and thus linear. So suppose that S is nonempty, with height h. Then we have that $\mathcal{I}(j,\, k) \subseteq S \cdot f$ for all $k < h$. Further, $S \cdot f$ contains no morphism $m : \mathcal{I}(j,\, k)$ whenever $k > h$, since $m \diamond f \notin S$ for any such m.

So it remains to show that $S \cdot f \cap \mathcal{I}(j,\, h)$ is a $<$-prefix of $\mathcal{I}(j,\, h)$. To do so, it's enough to show that if $m \in S \cdot f \cap \mathcal{I}(j,\, h)$ and $m' < m$ in $\mathcal{I}(j,\, h)$, then $m' \in S \cdot f$. Now from the premises and strict orientedness of $\mathcal{I}$ we have that $m \diamond f \in S$ and $m' \diamond f < m \diamond f$ in $\mathcal{I}(i,\, h)$. Since $S \cap \mathcal{I}(i,\, h)$ is a $<$-prefix of $\mathcal{I}(i,\, h)$, we deduce that $m' \diamond f \in S$ and $m' \in S \cdot f$. □

5.2.2∗6. The following lemma is an important property of linear cosieves; it is instructive to understand using the geometric intuition of (5.2.2∗3).

5.2.2∗7. Lemma. *Sufficiently small restrictions of linear cosieves are full.* Suppose that S is a linear cosieve under i in $\mathcal{I}$, of height h. Then for any $f : \mathcal{I}(i,\, j)$ with $j < h$, the restriction $S \cdot f$ is full.

5.3. SHAPES OF LINEAR COSIEVES

5.3∗1. Linear cosieves in countably simple semicategories have an alternative representation that is more conveniently suited to our goal of specifying matching objects. Specifically, there is a type which (1) faithfully presents the required structure of linear cosieves—in particular by satisfying refinements of (5.2.2∗5)—and (2) crucially, possesses a suitable well founded order over which we can induct. This is the type of *shapes* of linear cosieves.

5.3∗2. Definition. *Shapes of linear cosieves.* A *shape* of a linear cosieve in $\mathcal{I}$ is an element of the set

$$\mathrm{L}_{\mathcal{I}} :\equiv \sum \big((i, h, t) : \mathcal{I}_0 \times \mathcal{I}_0 \times \mathbb{N}\big),\ t \leq |\mathcal{I}(i,\, h)|.$$

We will usually abuse notation and denote a shape by its non-propositional components (i, h, t).

5.3∗3. Definition. *The linear cosieve presented by a shape.* A shape (i, h, t) presents a linear cosieve $S_{(i,h,t)}$ under i, which is defined to be the disjoint union of

- $\mathcal{I}(i,\, j)$ for all $j < h$ in $\mathcal{I}_0$, together with

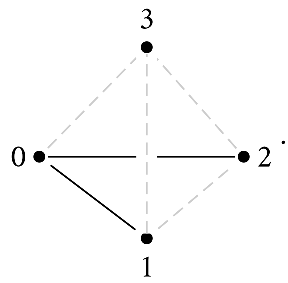
- the $<$-prefix of $\mathcal{I}(i,\, h)$ of length t.

This defines a *presentation map* $S_{(\_,\,\_,\,\_)} : (i, h, t) \mapsto S_{(i,\,h,\,t)}$.

5.3∗4. Example. *Linear cosieves from shapes.* Consider again the semicategory part of Δ_+^{op} as a countably simple semicategory ((5.1∗8) and (5.2.2∗2)). It's straightforward to check that $(3, 1, 2)$ is a shape in Δ_+^{op}, i.e. an element of $\mathrm{L}_{\Delta_+^{\mathrm{op}}}$.[2] Then

$$S_{(3,\,1,\,2)} \equiv \{(0)_0^3, (1)_0^3, (2)_0^3, (3)_0^3, (01)_1^3, (02)_1^3\},$$

which indexes the following 1-dimensional subskeleton of $\partial\Delta^3$:



Intuitively, a shape (i, h, t) in $\mathcal{I}$ presents a linear cosieve $S_{(i,\,h,\,t)}$ under $i : \mathcal{I}_0$ which indexes a "h-dimensional subskeleton of $\partial\bigcirc^i$", and which "contains the first t-many h-dimensional faces of $\partial\bigcirc^i$".

5.3∗5. *Linear cosieves are ordered.* Via the lexicographic order on $\mathrm{L}_{\mathcal{I}}$ and the presentation map, the subset order on linear cosieves under a fixed apex (5.2.2∗4) extends to an order on the entire type[3] of linear cosieves in $\mathcal{I}$.

5.3∗6. The presentation map is surjective but not injective—it is straightforward to check that shapes that are equivalent under the relation generated by

$$(i, h+1, 0) \sim (i, h, |\mathcal{I}(i,\, h)|)$$

are sent to equal cosieves. That is, $S_{(i,\,h+1,\,0)} = S_{(i,\,h,\,|\mathcal{I}(i,\,h)|)}$ for any $i, h : \mathcal{I}_0$.

5.4. SHAPE RESTRICTION

The properties of restriction of linear cosieves in strictly oriented semicategories $\mathcal{I}$ are crucial to the inductive construction of matching objects. Since we will use shapes in place of linear cosieves in our specification, we need to define a restriction operation on the type $\mathrm{L}_{\mathcal{I}}$ of shapes, and ensure that it is "natural" with respect to the presentation map $S_{(\_,\,\_,\,\_)}$ (5.3∗5). In particular, we need the analogue of (5.2.2∗7) for shapes. The purpose of this section is to do all this.

5.4∗1. Definition. *Factoring morphisms in countably simple semicategories.* Given morphisms $m : \mathcal{I}(i,\, h)$ and $f : \mathcal{I}(i,\, j)$, by enumerating $\mathcal{I}(j,\, h)$ we can decide

[2] Writing, as usual, i instead of $[i] : (\Delta_+^{\mathrm{op}})_0$.

[3] That is, taking the total space over apexes.

whether f divides m, i.e. whether there is a morphism $g : \mathcal{I}(j,\ h)$ such that $g \diamond f = m$. That is, there is a decidable relation

$$\_ \mid \_ : \prod (f : \mathcal{I}(i,\ j))\,(m : \mathcal{I}(i,\ h)) \sum (g : \mathcal{I}(j,\ h)),\ g \diamond f = m.$$

5.4∗2. Lemma. *Uniqueness of dividends.* When $\mathcal{I}$ is strictly oriented, (5.1∗9) implies that the set $\Sigma\,(g : \mathcal{I}(j,\ h))\,(g \diamond f = m)$ is a subsingleton for any $f : \mathcal{I}(i,\ j)$ and $m : \mathcal{I}(i,\ h)$. That is, whenever f divides m, the dividend g is unique.

5.4∗3. Now recall from (5.3∗3) that a shape $(i, h, t) : \mathrm{L}_{\mathcal{I}}$ describes a linear cosieve $S_{(i,\,h,\,t)}$ containing the first t-many morphisms of $\mathcal{I}(i,\ h)$. For $j : \mathcal{I}_0$ and $f : \mathcal{I}(i,\ j)$, we can recursively define a function that counts the number of these morphisms that factor through f, i.e. the number of indices $k < t$ such that $f \mid [k]^i_h$.

5.4∗4. Definition. Explicitly, define

$$\text{count-factors} : ((i, h, t) : \mathrm{L}_{\mathcal{I}})\ \{j : \mathcal{I}_0\}\ (f : \mathcal{I}(i,\ h)) \to \mathbb{N}$$

so that

$$\begin{aligned} \text{count-factors}\,(i, h, 0)\,f \quad &:\equiv\ 0 \\ \text{count-factors}\,(i, h, t+1)\,f \quad &:\equiv\ \text{let } n :\equiv \text{count-factors}\,(i, h, t)\,f \text{ in} \\ &\qquad \text{if } f \mid [t]^i_h \text{ then } (n+1) \text{ else } n \end{aligned}$$

5.4∗5. For the inductive specification of matching contexts to type-check, we require certain equalities to hold.

5.4∗6. Lemma. For any shape (i, h, t) and $f : \mathcal{I}(i,\ j)$ such that $j \le h$, we have

$$\text{count-factors}\,(i, h, t)\,f = 0.$$

Proof. By induction on t. True by definition for $t \equiv 0$. In the inductive case, $f \nmid [t]^i_h$ since $\mathcal{I}$ is inverse, and so $\text{count-factors}\,(i, h, t{+}1)\,f \equiv \text{count-factors}\,(i, h, t)\,f = 0$. □

5.4∗7. Theorem. *The factor theorem.* If $\mathcal{I}$ is strictly oriented, $i, j, h : \mathcal{I}_0$ and $f : \mathcal{I}(i,\ j)$, then

$$\text{count-factors}\,(i, h, |\mathcal{I}(i,\ h)|)\,f = |\mathcal{I}(j,\ h)|.$$

5.4.1. Proof of the factor theorem

5.4.1∗1. The analogue of (5.4∗7) in terms of linear cosieves is intuitively clear: it says that the disjoint union over $m : \mathcal{I}(i,\ h)$ of the factors of m through f is in bijection with $\mathcal{I}(j,\ h)$. This will yield the version of (5.2.2∗7) for shapes. To prove that it holds, we need a large number of technical intermediate results, which have also been formalized in Agda [Che24b].

5.4.1∗2. Assume that $\mathcal{I}$ is countably simple, $i, j, h : \mathcal{I}_0$, $f : \mathcal{I}(i,\ j)$, and that there is a smallest index $t_0 : \mathrm{Fin}|\mathcal{I}(i,\ h)|$ such that $f \mid [t_0]^i_h$. Then for all $t \le t_0$, $\text{count-factors}\,(i, h, t)\,f = 0$.

Proof. By induction on t. The equality holds by definition when $t = 0$. If $t+1 \le t_0$, then $f \nmid [t]^i_h$ and so

$$\text{count-factors}\,(i, h, t+1)\, f = \text{count-factors}\,(i, h, t)\, f = 0. \qquad \square$$

5.4.1∗3. Lemma. Assume that $\mathcal{I}$ is countably simple, $i, j, h : \mathcal{I}_0$ and $f : \mathcal{I}(i,\, j)$. The following are equivalent:

1. For all $t \le |\mathcal{I}(i,\, h)|$, count-factors $(i, h, t)\, f = 0$.
2. For all $t : \mathrm{Fin}|\mathcal{I}(i,\, h)|$, $f \nmid [t]^i_h$.
3. $|\mathcal{I}(j,\, h)| = 0$.

Proof. If $|\mathcal{I}(j,\, h)| > 0$, then $[0]^j_h$ exists. Let $k :\equiv \mathrm{idx}\,([0]^j_h \circ f)$. Then $k < |\mathcal{I}(i,\, h)|$ and $f \mid [k]^i_h$, and thus

$$\text{count-factors}\,(i, h, k+1)\, f = \text{count-factors}\,(i, h, k)\, f + 1 > 0.$$

This proves (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Finally, assume $|\mathcal{I}(j,\, h)| = 0$. By induction on t we show that (1) holds; in the inductive case, count-factors $(i, h, t+1)\, f >$ count-factors $(i, h, t)\, f = 0$ only if $f \mid [t]^i_h$. But this would give a factor $g : \mathcal{I}(j,\, h)$. $\square$

5.4.1∗4. Now assume that $\mathcal{I}$ is strictly oriented, $i, j, h : \mathcal{I}_0$, $f : \mathcal{I}(i,\, j)$, and $|\mathcal{I}(j,\, h)| > 0$. These assumptions also imply that $|\mathcal{I}(i,\, h)| > 0$ and, by (5.4.1∗3), that there is some $t : \mathrm{Fin}|\mathcal{I}(i,\, h)|$ such that $f \mid [t]^i_h$.

5.4.1∗5. Definition. We define "division by f",

$$[\_]/f : \mathrm{Fin}|\mathcal{I}(i,\, h)| \to \mathcal{I}(j,\, h),$$

by

$$\begin{aligned} [0]/f &:\equiv \text{ if } f \mid [0]^i_h \text{ with witness } g \text{ then } g \text{ else } [0]^j_h \\ [t+1]/f &:\equiv \text{ if } f \mid [t+1]^i_h \text{ with witness } g \text{ then } g \text{ else } [t]/f, \end{aligned}$$

and this function is well defined by (5.4∗2). Informally, $[t]/f$ is the least upper bound of the set of $g : \mathcal{I}(j,\, h)$ such that $g \circ f \le [t]^i_h$.

5.4.1∗6. Lemma. For all $t : \mathrm{Fin}|\mathcal{I}(i,\, h)|$ and $g : \mathcal{I}(j,\, h)$ such that $g \circ f \le [t]$, we have $g \le [t]/f$.

Proof. By induction on t. If $g \circ f \le [0]$, then $g \circ f = [0]$ and $g = [0]/f$. So assume $g \circ f \le [t+1]$. Suppose $[t+1] = g' \circ f$ for some g', then $g \circ f \le g' \circ f$ and so $g \le g' = [t+1]/f$. Otherwise $f \nmid [t+1]$, which implies $[t]/f = [t+1]/f$ and $g \circ f \le [t]$. Then by the induction hypothesis $g \le [t]/f = [t+1]/f$. $\square$

5.4.1∗7. Lemma. Let $t_0 : \mathrm{Fin}|\mathcal{I}(i,\, h)|$ be the smallest index for which $f \mid [t_0]^i_h$. For all $t : \mathrm{Fin}|\mathcal{I}(i,\, h)|$,

1. if $t \leq t_0$ then $[t]/f = [0]$, and
2. if $t \geq t_0$ then $([t]/f) \circ f \leq [t]$.

Proof.

1. The equality holds by induction on t for $t < t_0$. For $t = t_0$, we have $([t_0]/f) \circ f = [t_0]$ since $f \mid [t_0]$, and also $t_0 \leq \mathrm{idx}\,([0] \circ f)$ since $f \mid ([0] \circ f)$. Thus $([t_0]/f) \circ f = [t_0] \leq [0] \circ f$, and $[0] \leq [t_0]/f \leq [0]$ since $\mathcal{I}$ is strictly oriented.
2. By induction on $t \geq t_0$. We have that $([t_0]/f) \circ f = [t_0]$ since $f \mid [t_0]$. Assume $t + 1 > t_0$. If $f \mid [t+1]$, then $([t+1]/f) \circ f = [t+1]$. Otherwise $f \nmid [t+1]$, in which case

$$([t+1]/f) \circ f = ([t]/f) \circ f \leq [t] < [t+1]. \qquad \square$$

5.4.1∗8. Lemma. $[\_]/f$ is weakly monotone. That is, if $t, t' : \mathrm{Fin}|\mathcal{I}(i, h)|$ and $t < t'$, then $[t]/f \leq [t']/f$. Consequently, if $[t]/f < [t']/f$ then $t < t'$.

Proof. It's enough to show that $[t]/f \leq [t+1]/f$ for every $t + 1 : \mathrm{Fin}|\mathcal{I}(i, h)|$. Since we're working under the assumption that $|\mathcal{I}(j, h)| > 0$, by (5.4.1∗3) there is a smallest index $t_0 : \mathrm{Fin}|\mathcal{I}(i, h)|$ for which $f \mid [t_0]^i_h$. For every $t < t_0$, $[t]/f = [0]$ by definition. Assume $t \geq t_0$ such that $t + 1 : \mathrm{Fin}|\mathcal{I}(i, h)|$. By (5.4.1∗7),

$$([t]/f) \circ f \leq [t] < [t+1],$$

so $[t]/f \leq [t+1]/f$ by (5.4.1∗6). $\square$

5.4.1∗9. Since $[\mathrm{idx}\,(g \circ f)]/f = g$ for all $g : \mathcal{I}(j, h)$, we have that $[\_]/f$ surjects onto $\mathcal{I}(j, h)$.

5.4.1∗10. Corollary. Assume that $t + 1 : \mathrm{Fin}|\mathcal{I}(i, h)|$. Then

$$\mathrm{idx}\,([t]/f) \leq \mathrm{idx}\,([t+1]/f) \leq \mathrm{idx}\,([t]/f) + 1.$$

Moreover, the upper bound is an equality if $f \mid [t+1]^i_h$, there is a smallest index $t_0 : \mathrm{Fin}|\mathcal{I}(i, h)|$ such that $f \mid [t_0]^i_h$, and $t_0 \leq t$.

Proof. The lower bound holds by monotonicity.

Let $k :\equiv \mathrm{idx}\,([t]/f)$. If $k + 1 = |\mathcal{I}(j, h)|$ then the upper bound holds. So assume $k + 1 < |\mathcal{I}(j, h)|$, and let $[k+1] = [t']/f$ by surjectivity. If $\mathrm{idx}\,([t']/f) < \mathrm{idx}\,([t+1]/f)$, then

$$[t]/f = [k] < [k+1] = [t']/f < [t+1]/f,$$

and by (5.4.1∗8) we have $t < t' < t + 1$, a contradiction.

Now assume a smallest divisible index t_0 such that $t_0 \leq t$. If $f \mid [t+1]^i_h$ and $\mathrm{idx}\,([t]/f) = \mathrm{idx}\,([t+1]/f)$ then we get a contradiction

$$[t+1] = [t+1]/f \circ f = [t]/f \circ f \leq [t],$$

where the last inequality holds by (5.4.1∗7). $\square$

5.4.1∗11. Corollary. Let $|\mathcal{I}(i, h)| = T + 1$ and $|\mathcal{I}(j, h)| = J + 1$. Then

$$[T]/f = [J].$$

Proof. By surjectivity let $[J] = [t]/f$. Then by monotonicity,

$$[J] = [t]/f \leq [T]/f \leq [J]. \qquad \square$$

5.4.1∗12. The purpose of considering $[\_]/f$ is to prove that count-factors, which is defined with codomain $\mathbb{N}$, is in fact bounded by $|\mathcal{I}(j, h)|$. To do so, we relate the two functions by the following lemma.

5.4.1∗13. Lemma. Assume that $\mathcal{I}$ is strictly oriented, $i, j, h : \mathcal{I}_0$, $f : \mathcal{I}(i, j)$, and that there is a smallest index $t_0 : \mathrm{Fin}|\mathcal{I}(i, h)|$ such that $f \mid [t_0]^i_h$. Then for all $t_0 \leq t < |\mathcal{I}(i, h)|$,

$$\text{count-factors}\,(i, h, t + 1)\,f = \mathrm{idx}([t]/f) + 1.$$

Proof. When $t = t_0$, the equation holds by (5.4.1∗2) and (5.4.1∗7). Suppose the equality holds for $t \geq t_0$. Then count-factors $(i, h, t{+}2)\,f$ is equal to count-factors $(i, h, t{+}1)\,f$ if $f \nmid [t + 1]^i_h$, and to count-factors $(i, h, t + 1)\,f + 1$ if $f \mid [t + 1]^i_h$. In either case, this is equal to idx $([t + 1]/f) + 1$ by the induction hypothesis and (5.4.1∗10). $\square$

5.4.1∗14. Corollary. If $\mathcal{I}$ is strictly oriented, (i, h, t) is a shape and $f : \mathcal{I}(i, j)$, then

$$\text{count-factors}\,(i, h, t)\,f \leq |\mathcal{I}(j, h)|.$$

Proof. By (5.4.1∗3), if $|\mathcal{I}(j, h)| = 0$ then count-factors $(i, h, t)\,f = 0 \leq |\mathcal{I}(j, h)|$ for all t. Otherwise, there's a smallest index $t_0 < |\mathcal{I}(i, h)|$ such that $f \mid [t_0]^i_h$. Then for all $t \leq t_0$,

$$\text{count-factors}\,(i, h, t)\,f = 0$$

by (5.4.1∗2), while for $t_0 \leq t < |\mathcal{I}(i, h)|$,

$$\text{count-factors}\,(i, h, t + 1)\,f = \mathrm{idx}\,([t]/f) + 1 < |\mathcal{I}(j, h)| + 1$$

by (5.4.1∗13). Stated differently, for $t_0 < t \leq |\mathcal{I}(i, h)|$,

$$\text{count-factors}\,(i, h, t)\,f \leq |\mathcal{I}(j, h)|. \qquad \square$$

5.4.1∗15. With (5.4.1∗13) in hand we can finally show that (5.4∗7) holds.

5.4.1∗16. Proof. *Of the factor theorem (5.4∗7).* If $|\mathcal{I}(j, h)| = 0$, then by (5.4.1∗3) we also have count-factors $(i, h, |\mathcal{I}(i, h)|)\,f = 0$. Otherwise $|\mathcal{I}(j, h)| > 0$, thus also $|\mathcal{I}(i, h)| > 0$. So let $|\mathcal{I}(i, h)| = T + 1$ and $|\mathcal{I}(j, h)| = J + 1$, and by (5.4.1∗13) and (5.4.1∗11), we have that

$$\text{count-factors}\,(i, h, T{+}1)\,f = \mathrm{idx}\,([T]/f){+}1 = J{+}1. \qquad \square$$

5.4.1∗17. We also prove a few more results to be used in the construction of matching contexts.

5.4.1∗18. Lemma. Assume that $\mathcal{J}$ is strictly oriented, $i, h : \mathcal{J}_0$ are objects of $\mathcal{J}$, $t : \mathrm{Fin}|\mathcal{J}(i, h)|$ and $f : \mathcal{J}(i, j)$. If $f \mid [t]^i_h$, then

$$[t]/f = [\text{count-factors}\,(i, h, t)\, f]^j_h.$$

In particular, count-factors $(i, h, t)\, f < |\mathcal{J}(j, h)|$.

Proof. By (5.4.1∗13) and the definition of count-factors (5.4∗4),

$$\mathrm{idx}([t]/f) + 1 = \text{count-factors}\,(i, h, t + 1)\, f = \text{count-factors}\,(i, h, t)\, f + 1,$$

and so $[t]/f = [\text{count-factors}\,(i, h, t)\, f]^j_h$. □

5.4.1∗19. Lemma. Suppose $\mathcal{J}$ is strictly oriented, $i, j, k, h : \mathcal{J}_0$, $f : \mathcal{J}(i, j)$, $g : \mathcal{J}(j, k)$ and $t : \mathrm{Fin}|\mathcal{J}(i, h)|$. The following are equivalent:

1. $g \circ f \mid [t]^i_h$.
2. $f \mid [t]^i_h$ and $g \mid [\text{count-factors}\,(i, h, t)\, f]^j_h$.

Proof. For convenience, abbreviate $c :\equiv$ count-factors $(i, h, t)\, f$. Assume that $[t]^i_h = m \circ (g \circ f)$ for some $m : \mathcal{J}(k, h)$. Then $f \mid [t]^i_h$ with $[t]/f = m \circ g$. But by (5.4.1∗18), $m \circ g = [t]/f = [c]^j_h$, and so $g \mid [c]^j_h$.

Conversely, if $f \mid [t]^i_h$ then $[t]^i_h = [t]/f \circ f = [c]^j_h \circ f$ by (5.4.1∗18). Thus if $[c]^j_h = m \circ g$ for some m then $[t]^i_h = m \circ g \circ f$. □

5.4.1∗20. Corollary. If $\mathcal{J}$ is strictly oriented, then for any $f : \mathcal{J}(i, j)$, $g : \mathcal{J}(j, k)$, $h : \mathcal{J}_0$ and $t : \mathrm{Fin}|\mathcal{J}(i, h)|$, we have

$$\text{count-factors}\,(i, h, t)\,(g \diamond f) = \text{count-factors}\,(j, h, (\text{count-factors}\,(i, h, t)\, f))\, g.$$

5.4.2. The restriction operation on shapes

Now we can finally define the restriction on shapes of linear cosieves.

5.4.2∗1. Definition. *Shape restriction.* In a strictly oriented countably simple semicategory $\mathcal{J}$, the restriction operation on shapes

$$\_ \cdot \_ : ((i, h, t) : \mathrm{L}_{\mathcal{J}})\, \{j : \mathcal{J}_0\}\, (f : \mathcal{J}(i, j)) \to \mathrm{L}_{\mathcal{J}}$$

is defined by

$$(i, h, t) \cdot f :\equiv (j, h, \text{count-factors}\,(i, h, t)\, f).$$

By (5.4.1∗14), this again yields a shape.

5.4.2∗2. We then have the analogue of (5.2.2∗7) for shapes.

5.4.2∗3. Corollary. By (5.4∗6), if $(i, h, t) : \mathrm{L}_{\mathcal{I}}$ is a shape in a strictly oriented I and $f : \mathcal{I}(i,\ j)$ with $j \leq h$, then

$$(i, h, t) \cdot f = (j, h, 0).$$

By (5.3∗6), the shape $(j, h, 0)$ presents the full cosieve under j.

5.4.2∗4. Lemma. By (5.4.1∗20), if $\mathcal{I}$ is strictly oriented, then for any $f : \mathcal{I}(i,\ j)$, $g : \mathcal{I}(j,\ k)$, $h : \mathcal{I}_0$ and $t : \mathrm{Fin}|\mathcal{I}(i,\ h)|$,

$$(i, h, t) \cdot (g \diamond f) = (i, h, t) \cdot f \cdot g.$$

Chapter 6
Internal Classifiers of Reedy-fibrant $\mathcal{I}$-Types

This chapter contains joint work with Nicolai Kraus. Throughout, we assume a strictly oriented countably simple semicategory $\mathcal{I}$, and a 2-coherent wild cwf $\mathscr{C}$ (3.3∗13) with a Π-structure (3.5∗1) and a partial universe structure (3.5∗4).

6.1. OVERVIEW AND AIMS

6.1∗1. We aim to use the theory developed in this thesis to construct classifiers of Reedy-fibrant inverse diagrams from $\mathcal{I}$ to "sufficiently well behaved" $\mathscr{C}$. The macrostructure of our construction is to broadly follow the construction (4.4∗9) and (4.4∗15) described in Section 4.4, with two very important distinctions:

1. In contrast to the setting of Section 4.4, our construction should be completely HoTT-internal. That is, we aim for HoTT to take the place of 2LTT in (4.4∗9), and the 2-coherent internal model $\mathscr{C}$ (with $\hat{\Pi}$ and $\mathbb{U}$) the place of the fibrant universe Type^+.
2. Additionally, in plain HoTT we have no recourse to the theorem (4.4∗11), and therefore have to construct the matching objects simultaneously by hand. We aim to do so using the theory of shapes of linear cosieves in $\mathcal{I}$ developed in Chapter 5, as well as the coherence conditions on $\mathscr{C}$ studied in Chapter 3.

6.1∗2. Warning. As of the writing of this thesis, this construction is still only a sketch; its formalization in Agda [Che24a] is incomplete due to issues with termination and definitional equality. It may thus instead be thought of as a "specification" for the type of Reedy-fibrant $\mathcal{I}$-types.

6.1∗3. We do *not* expect our construction, when completed, to yield a general definition of internal classifiers of *type-valued* inverse diagrams[1] in HoTT—and in particular, we do not intend to solve the problem of defining semisimplicial types in plain HoTT.

6.1∗4. Instead, one of our more concrete goals is to give a precise internal statement of the folkloric idea that "if HoTT can interpret itself, then semisimplicial types can be constructed in HoTT" [Shu14]. First, observe that:

[1] i.e. diagrams into the universe type $\mathscr{U}$.

1. we can expect our construction to go through in the case that $\mathscr{C}$ is *set-level*, and in particular when $\mathscr{C}$ = Syn is the syntax QIIT of Altenkirch and Kaposi [AK16],[2] and that
2. the type of morphisms $[\![\_]\!] : \mathrm{2CohCwf}_{\hat{\Pi},\mathbb{U}}(\mathrm{Syn}, \mathcal{U})$ of 2-coherent wild cwfs with Π- and partial universe structure is internally definable in the standard way from the generalized algebraic definitions, and its elements may be considered *interpretations* of HoTT in itself.

We can then expect to transfer the construction in Syn along any element $[\![\_]\!]$ of $\mathrm{2CohCwf}_{\hat{\Pi},\mathbb{U}}(\mathrm{Syn}, \mathcal{U})$, thereby transforming the problem of constructing HoTT-internal Reedy-fibrant classifiers into that of constructing such interpretations, i.e. of inhabiting the type $\mathrm{2CohCwf}_{\hat{\Pi},\mathbb{U}}(\mathrm{Syn}, \mathcal{U})$.

6.1∗5. Additionally, we believe our proposal to be a "morally correct" general construction of internal classifiers, and intend to use obstructions to this construction as a source of inspiration for internal coherence conditions to investigate further. Significantly, in our setting these coherence conditions may now be *on the notion of internal model that we use*, and in this way we situate the problem of constructing semisimplicial types and other Reedy-fibrant diagram classifiers in HoTT in the larger setting of studying good notions of internal model of HoTT.

6.2. SPECIFICATION OF CLASSIFIERS OF REEDY-FIBRANT $\mathcal{I}$-TYPES

6.2∗1. *Data of the classifier.* By analogy with (4.4∗15), we mutually declare a function

$$\mathbb{R} : \mathcal{I}_0 \to \mathscr{C}_0$$

together with functions defining what we call the *partial matching objects*,

$$\begin{aligned}
\mathrm{M} &: \big((i,h,t) : \mathrm{L}_{\mathcal{I}}\big) \to \mathsf{Tel}\,(\mathbb{R}_{h+1}) \\
\mathrm{M}^{\to} &: \big((i,h,t) : \mathrm{L}_{\mathcal{I}}\big)\ (f : \mathcal{I}(i,\ j)) \to \mathscr{C}(\overline{\mathrm{M}_{(i,h,t)}},\ \overline{\mathrm{M}_{(i,h,t)\cdot f}}) \\
\mathrm{M}^{\diamond} &: \big((i,h,t) : \mathrm{L}_{\mathcal{I}}\big)\ (f : \mathcal{I}(i,\ j))\ (g : \mathcal{I}(j,\ k)) \\
&\quad \to \mathrm{M}^{\to}_{(j,h,\cdots)} g \diamond \mathrm{M}^{\to}_{(i,h,t)} f = \mathrm{idd}(\mathrm{ap}\,\mathrm{M}\,e) \diamond \mathrm{M}^{\to}_{(i,h,t)}(g \diamond f)
\end{aligned}$$

These give the data of the matching objects of elements of $\mathbb{R}_i$, in a way which we shortly describe.

Here, e is the witness of the equality $(i,h,t) \cdot (g \diamond f) = (i,h,t) \cdot f \cdot g$ given by (5.4.2∗4), without which $\mathrm{M}^{\diamond}$ would be ill typed. We have also implicitly quantified over objects $j, k : \mathcal{I}_0$ where necessary.

6.2∗2. Warning. For presentation reasons we will completely elide the witnesses that the indices (i,h,t) in our inductive specifications define shapes (i.e. that $t \leq |\mathcal{I}(i,\ h)|$). Even though these witnesses are propositional, they still impose *definitional* equality constraints that must be satisfied in order for our proposed

[2] Syn is indeed a wild cwf with Π- and partial universe structure.

definitions to type-check. This can be arranged, with difficulty, to some extent, but is part of the obstruction to the completion of the full definition.

6.2∗3. *The total matching object.* The data of the partial matching objects given by M, $\mathrm{M}^{\rightarrow}$ and $\mathrm{M}^{\diamond}$ are a refinement of the matching objects of Reedy-fibrant diagrams. Having declared M, we can define the actual *total* matching object $\mathrm{M}^{\mathrm{tot}}_i : \mathsf{Tel}\,\mathbb{R}_i$ of dimension i as

$$\begin{aligned} \mathrm{M}^{\mathrm{tot}}_0 &:\equiv \bullet \\ \mathrm{M}^{\mathrm{tot}}_{i+1} &:\equiv \mathrm{M}_{(i+1,i,|\mathcal{I}(i+1,i)|)} \end{aligned}$$

6.2∗4. *Specification of* $\mathbb{R}$. By analogy with (4.4∗9) we then define, by induction on $\mathcal{I}_0$,

$$\begin{aligned} \mathbb{R}_0 &:\equiv \blacklozenge \\ \mathbb{R}_{i+1} &:\equiv \mathbb{R}_i.\mathbb{A}_i, \end{aligned}$$

where $\mathbb{A}_i$ abbreviates $\hat{\Pi}\,\mathrm{M}^{\mathrm{tot}}_i\,\mathbb{U}$. Note that $\mathbb{A}_0 \equiv \mathbb{U}$ by definition (3.6∗10).

6.3. PARTIAL MATCHING OBJECTS

6.3∗1. *The generic* $\mathrm{M}^{\mathrm{tot}}_i$*-indexed type.* By (3.6∗13), there is a generic $\mathrm{M}^{\mathrm{tot}}i$-indexed type

$$A_i : \mathsf{Ty}\,(\overline{\mathrm{M}^{\mathrm{tot}}_i[\mathsf{p}_{\mathbb{A}_i}]})$$

for each $i : \mathcal{I}_0$.

6.3∗2. *The partial matching object—telescope part.* We specify M exhaustively for all appropriate i, h and t by

$$\begin{aligned} \mathrm{M}_{(i,0,0)} &:\equiv \bullet \\ \mathrm{M}_{(i,h+1,0)} &:\equiv \mathrm{M}_{(i,h,|\mathcal{I}(i,h)|)}[\mathsf{p}_{\mathbb{A}_{h+1}}] \\ \mathrm{M}_{(i,h,t+1)} &:\equiv \mathrm{M}_{(i,h,|\mathcal{I}(i,h)|)} \blacktriangleright A_h[\mathrm{idd}(e) \diamond \mathrm{M}^{\rightarrow}_{(i,h,t)}[t]^i_h]_{\mathsf{T}} \end{aligned}$$

Here, e is an equality built from (5.4∗6) and the following equality that must be proven mutually yet again:

6.3∗3. Lemma. By induction on $h : \mathcal{I}_0$, we have that

$$\mathrm{M}^{\mathrm{tot}}_h[\mathsf{p}_{\mathbb{A}_h}] = \mathrm{M}_{(h,h,0)}.$$

6.3∗4. The specification given in (6.3∗2) is not a proper definition, as we have not yet articulated an *internal* induction scheme that is compatible with the other components of the mutual declaration. Attempting to use (6.3∗2) as a definition in Agda together with the other specifications given below results in errors in termination checking.

6.3∗5. *Partial matching object—morphism part.* Now we consider the morphism action of the partial matching object. Let $f : \mathcal{I}(i, j)$, and consider the following two cases for appropriate i, h and t, ignoring the $(i, h, t+1)$ case.

- For $\mathrm{M}^{\rightarrow}_{(i,0,0)}f$, we compute that the type $\mathscr{C}(\overline{\mathrm{M}_{(i,0,0)}}, \overline{\mathrm{M}_{(i,0,0)\cdot f}})$ is definitionally equal to $\mathscr{C}(\mathbb{R}_1, \mathbb{R}_1)$, and define

$$\mathrm{M}^{\rightarrow}_{(i,0,0)}f :\equiv \mathrm{id}.$$

- Similarly, the type of $\mathrm{M}^{\rightarrow}_{(i,h+1,0)}f$ is definitionally equal to

$$\mathscr{C}(\overline{\mathrm{M}_{(i,h,|\mathcal{I}(i,h)|)}[\mathrm{p}_{\mathbb{A}_{h+1}}]}, \overline{\mathrm{M}_{(j,h,|\mathcal{I}(j,h)|)}[\mathrm{p}_{\mathbb{A}_{h+1}}]}).$$

  We therefore specify $\mathrm{M}^{\rightarrow}_{(i,h+1,0)}f$ to be the weakening (3.6∗9) of a certain substitution from $\overline{\mathrm{M}_{(i,h,|\mathcal{I}(i,h)|)}}$ to $\overline{\mathrm{M}_{(j,h,|\mathcal{I}(j,h)|)}}$ constructed “inductively” using

$$\mathrm{M}^{\rightarrow}_{(i,h,|\mathcal{I}(i,h)|)} : \mathscr{C}(\overline{\mathrm{M}_{(i,h,|\mathcal{I}(i,h)|)}}, \overline{\mathrm{M}_{(i,h,|\mathcal{I}(i,h)|)\cdot f}}).$$

  However, the weakening (3.6∗9) creates yet another proof obligation, due to its premise that a certain triangle of substitutions commutes.

6.3∗6. As we see, our attempts to define even the beginning of the partial matching object appears to run into ever increasing proof obligations. This is not quite the same as, but somewhat reminiscent of the familiar problem of infinite coherence in e.g. attempts to define semisimplicial types. Initially, I expected to be able to discharge these proof burdens by assuming that the internal wild cwf was in fact set-level, but have not been able to make this and other approaches work as of the writing of this thesis.

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