2-Dimensional

Internal Homotopical Type Theory

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I. *Model Theory of MLTT*

Martin-Löf type theory provides a formal logical basis for speaking about

- mathematical ***objects***

  "*Singular cohomology theory* $H^\bullet(X; R, A)$"

- their ontology—what ***type*** of thing they are

  "*...is a certain cochain complex of R-modules such that...*"

- in ***contexts*** of other appropriately typed objects

  $H^\bullet(X; R, A)$ *is defined for a space X, ring R and R-module A*

- where we can ***substitute*** with objects defined in ***other contexts***

  "*Let $n \in \mathbb{N}$, $R = \mathbb{Z}$, A be a $\mathbb{Z}$-module and $X = K(n, A)$.*
  Then $H^n(K(n, A); \mathbb{Z}, A) \cong \hom(A, A)$."

As with any formal logic, have notion(s) of ***model***.

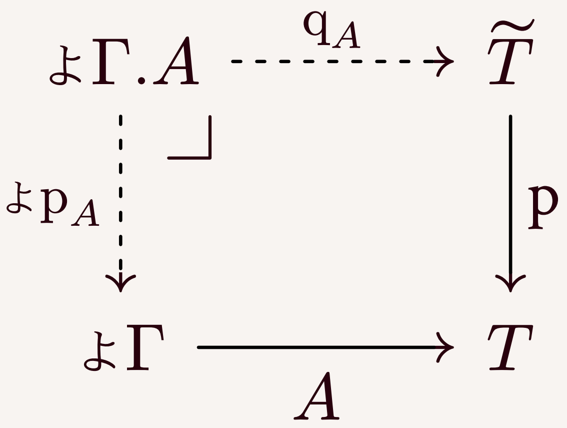
Clans/Display map cats, Comprehension cats, CwAs, CwFs/Natural models, …

(Fibration cats, Path cats, Model cats, …)

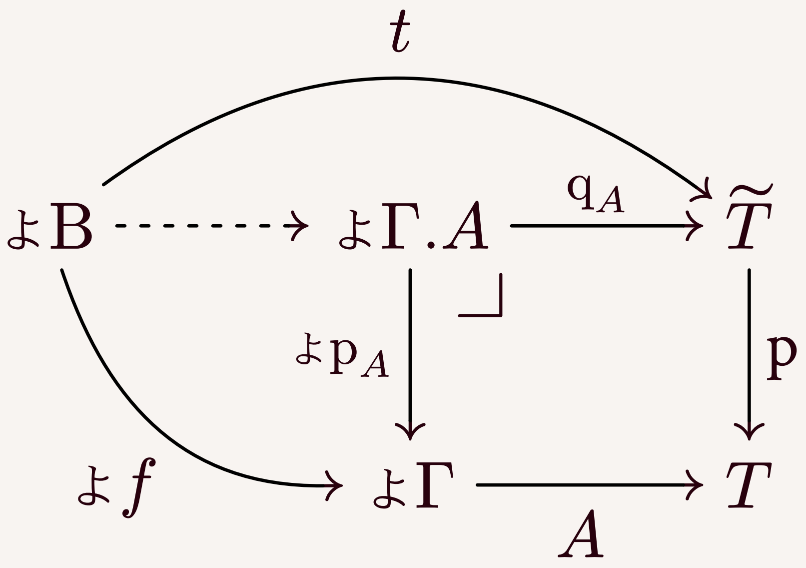
Common structural feature:

1. Underlying ***category*** $\mathcal{C}$ ***of contexts*** $\Gamma \in \mathcal{C}_0$ ***and substitutions*** $f \in \mathcal{C}(\Gamma, \Delta)$,
2. over which lie the structure of ***types*** and ***terms***,
3. which, in turn, ***extend*** contexts and substitutions.

Natural models

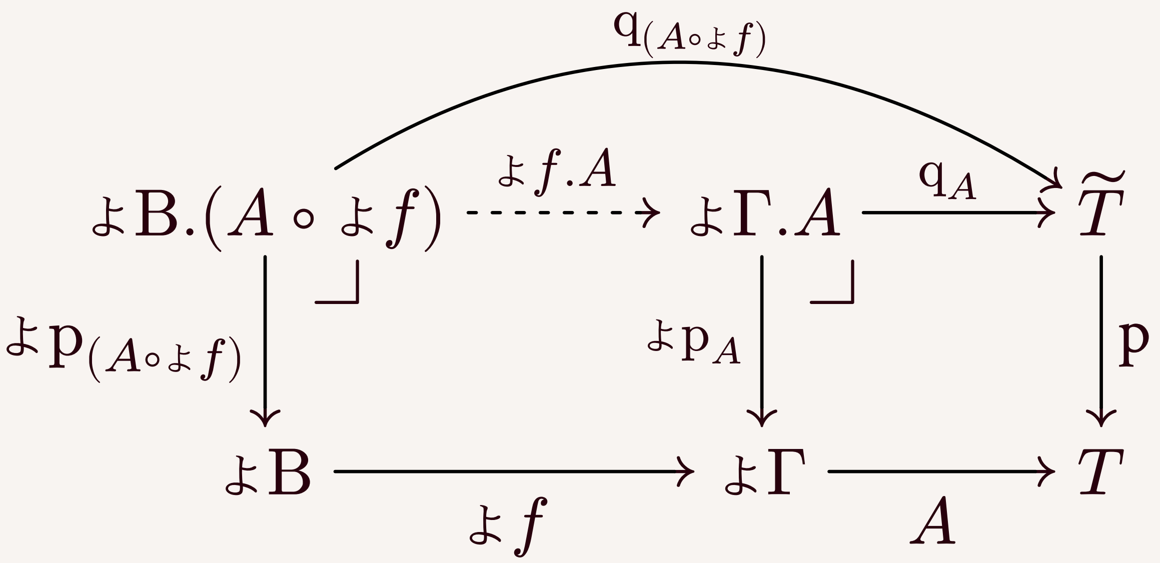


1. $\mathcal{C}$ category of contexts
2. $T, \widetilde{T} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$
3. $\mathrm{p} : \widetilde{T} \Rightarrow T$ ***representable*** *(algebraically)*



Universal property: *Extension of substitutions.*

Substitutions $\mathcal{C}(\mathrm{B}, \Gamma.A)$ ***are uniquely determined by substitutions*** $f : \mathcal{C}(\mathrm{B}, \Gamma)$ ***and terms*** t ***of type*** $(A \circ よ f)$.



Consequence: *Substitution in types induces pullbacks of contexts.*

Every "fibration" $\mathrm{p}_A \in \mathcal{C}(\Gamma.A, \Gamma)$ ***has chosen pullbacks along arbitrary morphisms*** $f \in \mathcal{C}(\mathrm{B}, \Gamma)$.

II. *Internal Type Theory*

Internal type theory

Develop the categorical model theory of type theory in type theory itself.

(Dybjer, '96)

Study the theory of models in the abstract, and
construct concrete instances of such models,
entirely using the logic and language of MLTT.

Notion of model used by *op. cit.* : ***categories with families***.

Categories with families

Algebraic presentation of natural models.

1. $\mathcal{C}$ category
2. $\mathrm{Ty} : \mathcal{C}^{\mathrm{op}} \to \mathcal{U}, \;\; \mathrm{Tm} : \left(\int_{\mathcal{C}} \mathrm{Ty}\right)^{\mathrm{op}} \to \mathcal{U}$
3. *Functorial choice of representing objects for the presheaves*

$$\left(\mathrm{B} \xrightarrow{f} \Gamma\right) \longmapsto \mathrm{Tm}_{\mathrm{B}}(A[f]) \; : \; (\mathcal{C}/\Gamma)^{\mathrm{op}} \to \mathcal{U}$$

indexed over $\Gamma \in \mathcal{C}_0$ *and* $A \in \mathrm{Ty}_{\Gamma}$.

Categories with families

Algebraic presentation of natural models.

Data:

Contexts	Substitutions		
$\Gamma, \Delta : \mathcal{C}_0$	$f : \mathcal{C}(\Gamma, \Delta)$		
Types	*Substitution in types*	*Terms*	*Substitution in terms*
$\mathrm{Ty}(\Delta) : \mathcal{U}$, $A : \mathrm{Ty}(\Delta)$	$A[f] : \mathrm{Ty}(\Gamma)$	$\mathrm{Tm}_\Delta(A) : \mathcal{U}$, $a : \mathrm{Tm}_\Delta(A)$	$a[f] : \mathrm{Tm}_\Gamma(A[f])$
Context extension	*Display map*	*Fresh variable*	*Universal map*
$\Delta.A : \mathcal{C}_0$	$\mathrm{p}_A : \mathcal{C}(\Delta.A, \Delta)$	$\mathrm{q}_A : \mathrm{Tm}(A[\mathrm{p}_A])$	$(f, t) : \mathcal{C}(\Gamma, \Delta.A)$

Categories with families

Algebraic presentation of natural models.

Equations:

Functoriality

$$[\mathrm{id}] : A[\mathrm{id}] = A \qquad [\circ] : A[f \circ g] = A[f][g]$$

$$[\mathrm{id}]_{\mathrm{t}} : a[\mathrm{id}] \overset{*}{=} a \qquad [\circ]_{\mathrm{t}} : a[f \circ g] \overset{*}{=} a[f][g]$$

Representability

$$\mathrm{p}\beta : \mathrm{p}_A \circ (f, t) = f \qquad \mathrm{q}\beta : \mathrm{q}[(f, t)] \overset{*}{=} t$$

$$\eta : (\mathrm{p}_A, \mathrm{q}_A) = \mathrm{id} \qquad \mathrm{dist} : (f, t) \circ g = (f \circ g, t[g]_*)$$

Taking the preceding algebraic signature we get a good internal definition of "model of MLTT", in MLTT *with uniqueness of identity proofs* (UIP).

In particular: the universe type $\mathcal{U}$ is a category and can be equipped with cwf structure—the *standard/metacircular model*. This is an "inner model" of MLTT with UIP.

III. *Internal Homotopical Type Theory*

In MLTT *without* UIP, taking the preceding signature of a cwf as a definition of "internal model" is undesirable:

- Does not include *higher dimensional examples.*

  (*Category of contexts should not be restricted to (1-)categories.*)

- Prototypical *inner model* $\mathcal{U}$ is not an instance.

  (*$\mathcal{U}$ is not a (1-)category in higher MLTT.*)

...So just define a higher/∞-categorical version of cwfs?

But defining ∞-categories is one of the last major open problems in HoTT/UF.

Instead...

Definition. A ***wild category*** $\mathcal{C}$ consists of

- $\mathcal{C}_0 : \mathcal{U}$
- $\mathcal{C}(x, y) : \mathcal{U}$ for all $x, y : \mathcal{C}_0$
- id, $\circ$ as usual

and *unitors and associators*

$$\rho : g \circ \mathrm{id} = g, \quad \lambda : \mathrm{id} \circ f = f, \quad \alpha : (f \circ g) \circ h = f \circ g \circ h.$$

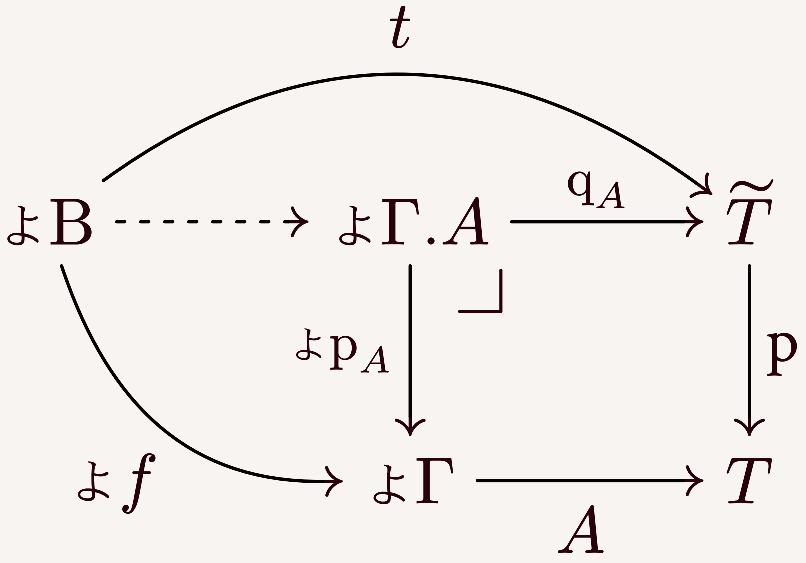
These equalities are **not** propositions!

Definition. A *wild cwf* is a model of the signature of a cwf, but with an underlying *wild* category of contexts $\mathcal{C}$.

How good a notion of internal model of homotopical type theory is this?

Almost trivially: the universe type $\mathcal{U}$ in MLTT without UIP is a wild cwf.

Recall the universal property of representable pullback squares of natural models,



Substitutions $\mathcal{C}(\Gamma, \Delta.A)$ *are uniquely determined by substitutions* $f : \mathcal{C}(\Gamma, \Delta)$ *and terms* t *of type* $A[f]$.

The same universal property holds in wild cwfs.

Proposition. *If $\mathcal{C}$ is a wild cwf with $\Gamma, \Delta : \mathcal{C}_0$ and $A : \mathrm{Ty}(\Delta)$, then there is an equivalence of types*

$$\mathcal{C}(\Gamma, \Delta.A) \simeq (f : \mathcal{C}(\Gamma, \Delta)) \times \mathrm{Tm}(A[f])$$

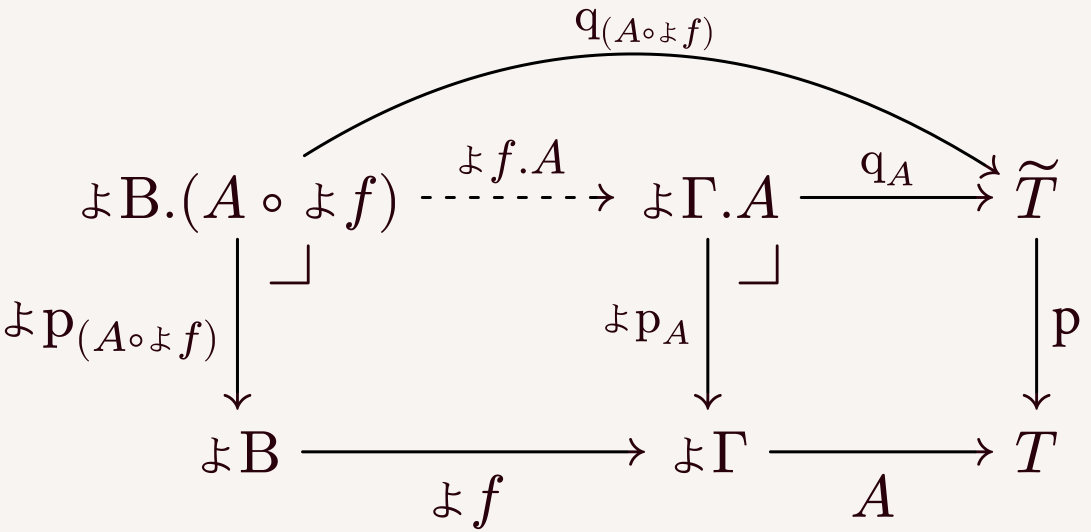
whose forward map is

$$f \mapsto (\mathrm{p}_A \circ f, \mathrm{q}_A[f]_*).$$

Corollary. *Suppose that* $\mathcal{C}$ *is a wild cwf. For any* $\Gamma : \mathcal{C}_0$ *and* $A : \mathrm{Ty}(\Gamma)$, *there is an equivalence of types between* $\mathrm{Tm}(A)$ *and the type of sections of* p_A *in* $\mathcal{C}$.

i.e. "In any suitable notion of higher model", the terms of A and the sections of p_A have the same homotopy type.

Recall that fibrations of natural models have pullbacks along arbitrary $\mathcal{C}$-morphisms,

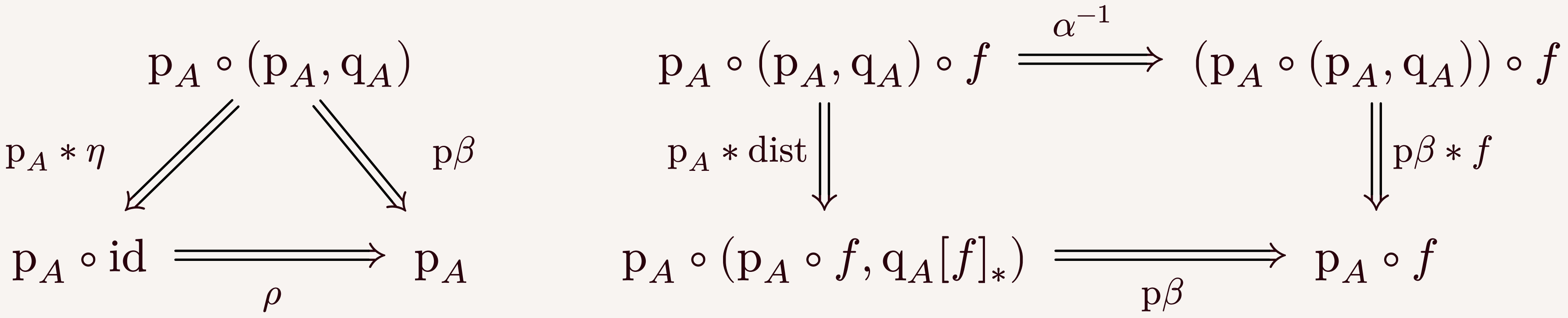


Every "fibration" $p_A \in \mathcal{C}(\Gamma.A, \Gamma)$ *has chosen pullbacks along arbitrary morphisms* $f \in \mathcal{C}(\mathrm{B}, \Gamma)$.

An arbitrary wild cwf does* not *satisfy this property!

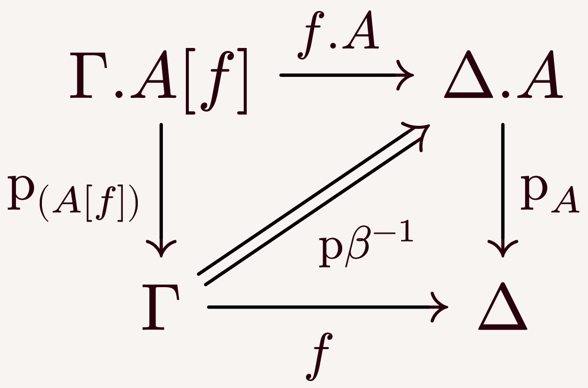
Need *2-coherence*:

- $\mathcal{C}$ must satisfy bicategorical 2-coherence axioms,
- Ty must satisfy pseudofunctorial 2-coherence axioms, and
- for all $\Gamma, \Delta : \mathcal{C}_0$, $A : \mathrm{Ty}(\Delta)$ and $f : \mathcal{C}(\Gamma, \Delta.A)$, there should be "3-cells" witnessing commutativity of the diagrams of equalities of $\mathcal{C}$-morphisms:



Definition. A wild cwf that satisfies the conditions of the previous slide is called *2-coherent*.

Theorem. *For any* 2-coherent wild cwf $\mathcal{C}$, $A : \mathrm{Ty}(\Delta)$ *and* $f : \mathcal{C}(\Gamma, \Delta)$, *the square*



where $f.A :\equiv \left(f \circ \mathrm{p}_{(A[f])}, \mathrm{q}_{(A[f])_*}\right)$, *is a* **pullback** *in* $\mathcal{C}$.

Examples of 2-coherent wild cwfs:

- Any 1-categorical cwf, e.g. the *syntax QIIT* (Altenkirch, Kaposi '16)
- The *universe type* in homotopical type theory, also *any subuniverse*
- The *higher container model* (Damato, Altenkirch, Kaposi)

Coherent "splitness"

In 2-coherent wild cwfs, functoriality of Ty extends to the chosen pullbacks:

Proposition. *In any 2-coherent wild cwf $\mathcal{C}$, we have equalities of pullbacks:*

$$\begin{array}{ccc} \Gamma.A[\mathrm{id}] & \xrightarrow{\mathrm{id}\,.A} & \Gamma.A \\ {\scriptstyle \mathrm{p}_{(A[\mathrm{id}])}}\downarrow & \overset{\mathrm{p}\beta^{-1}}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle \mathrm{p}_A} \\ \Gamma & \xrightarrow[\mathrm{id}]{} & \Gamma \end{array} \quad = \quad \begin{array}{ccc} \Gamma.A & \xrightarrow{\mathrm{id}} & \Gamma.A \\ {\scriptstyle \mathrm{p}_A}\downarrow & \overset{\lambda\cdot\rho^{-1}}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle \mathrm{p}_A} \\ \Gamma & \xrightarrow[\mathrm{id}]{} & \Gamma \end{array}$$

$$\begin{array}{ccccc} \mathrm{B}.A[f][g] & \xrightarrow{g.A[f]} & \Gamma.A[f] & \xrightarrow{f.A} & \Delta.A \\ {\scriptstyle \mathrm{p}_{(A[f][g])}}\downarrow & \overset{\mathrm{p}\beta^{-1}}{\nearrow\!\!\!\!\nearrow} & {\scriptstyle \mathrm{p}_{(A[f])}}\downarrow & \overset{\mathrm{p}\beta^{-1}}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle \mathrm{p}_A} \\ \mathrm{B} & \xrightarrow[g]{} & \Gamma & \xrightarrow[f]{} & \Delta \end{array} \quad = \quad \begin{array}{ccc} \mathrm{B}.A[f\circ g] & \xrightarrow{(f\circ g).A} & \Delta.A \\ {\scriptstyle \mathrm{p}_{(A[f\circ g])}}\downarrow & \overset{\mathrm{p}\beta^{-1}}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle \mathrm{p}_A} \\ \mathrm{B} & \xrightarrow[f\circ g]{} & \Delta \end{array}$$

When $\mathcal{C}$ is additionally **univalent**, *these equality types are contractible.*

2-Coherent Internal Models of Homotopical Type Theory

arxiv.org/abs/2503.05790