Pullbacks and coherences for wild cwfs

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What I'm working on

Motivation: to construct...

- certain functors (Reedy fibrant inverse diagrams),
- into certain categorical structures (sufficiently coherent wild categories with families),

(similar: §4.5.5.2 of (Kolomatskaia-Shulman '24))

- in HoTT (for reasons).

But you can understand the rest of the talk independently of the above.

Background to this talk

Model theory of MLTT without K

- Homotopical/∞-categorical models
  (Awodey, Warren, Voevodsky, Kapulkin, Lumsdaine, Shulman, Uemura, ...)
- Internal models
  (Dybjer, Chapman, Altenkirch, Kaposi, ...)

Background to this talk

Model theory of MLTT without K

- Homotopical/∞-categorical models — not internal
  (Awodey, Warren, Voevodsky, Kapulkin, Lumsdaine, Shulman, Uemura, ...)
- Internal models — not homotopical
  (Dybjer, Chapman, Altenkirch, Kaposi, ...)

Roadmap

Looking for an "internal model theory" of homotopical type theory: Notion of higher model of HoTT *in HoTT*.

Main desiderata

Should include the syntax and the universe models (the paradigmatic set-based and ∞-models).

Main obstacle

Models of HoTT are ∞-toposes, the universe is an $(\infty, 1)$-category, and we don't know how to define $(\infty, 1)$-categories in plain HoTT.

Roadmap

Detour and try to find our way back to a solution:
Wild categories with coherence.

Fruit to pick on the way: HoTT-internal notion of "$(n, 1)$-categorical model of type theory"?
(i.e. morphisms form $(n-1)$-types)

In this talk

In plain HoTT,

- Wild categories and pullbacks (Part 1)
- Wild categories with families (Part 2)
- Coherence conditions for each (throughout)

Part 1

Wild categories

Don't know how to do ∞-categories, but still want to manipulate structure that *should* be ∞-categorical, in plain HoTT.

Stepping stone solution: *wild* categories.

Wild categories

Wild category = "precategory with arbitrary hom-types".

Definition

A *wild category* $\mathscr{C}$ consists of

- $\mathscr{C}_0$: Type
- $\mathscr{C}(x, y)$: Type for all $x, y : \mathscr{C}_0$
- id, $\_ \circ \_$ as usual

and *unitors and associators:*

- $\rho : g \circ \mathrm{id} = g$
- $\lambda : \mathrm{id} \circ f = f$
- $\alpha : (f \circ g) \circ h = f \circ g \circ h$

These equality types are not propositions!

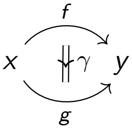
Wild categories

Examples

- Any precategory (hence any strict category or univalent 1-category).
- Any universe type $\mathcal{U}$:
  - $\mathcal{U}_0 :\equiv \mathcal{U}$
  - $\mathcal{U}(A, B) :\equiv A \to B$
  - id, composition as for functions
  - ρ, λ, α all refl.

Wild categories

Diagrammatic notation:



objects	$x, y : \mathscr{C}_0$
morphisms (1-cells)	$f, g : \mathscr{C}(x,\ y)$
equalities of morphisms (2-cells)	$\gamma : f = g$
higher equalities (higher cells)	$\ldots$

Whiskering

Given

$$x \xrightarrow{f} y \underset{h}{\overset{g}{\rightrightarrows}} z \;(\Downarrow \gamma), \qquad \gamma : g = h$$

have

$$\gamma * f : g \circ f = h \circ f$$
$$\gamma * f :\equiv \mathsf{ap}\,(\_ \circ f)\,\gamma$$

Similar for left whiskering.

Properties of whiskering

Satisfies equations

$$\mathsf{refl} * f = \mathsf{refl}$$
$$(\gamma * f)^{-1} = \gamma^{-1} * f$$
$$\gamma * \mathsf{id} = \rho \cdot \gamma \cdot \rho^{-1}$$

etc. by induction.

Equivalence and univalence

Definition

A morphism $f : \mathscr{C}(x, y)$ is a *wild $\mathscr{C}$-equivalence* if it is biinvertible in $\mathscr{C}$, i.e. has a section s and a retraction r so that

$$f \circ s = \mathrm{id} \quad \text{and} \quad r \circ f = \mathrm{id}$$

Write $x \simeq_{\mathscr{C}} y$ if there is a wild equivalence $f : \mathscr{C}(x, y)$.

Equivalence and univalence

Notation

Instead of transport(P, p, x) we write

$$x \downarrow_p^P$$

Definition

For any equal objects $e : x = y$ of a wild category $\mathscr{C}$, there is a "dependent identity" morphism

$$\mathrm{idd}(e) : \mathscr{C}(x, y)$$
$$\mathrm{idd}(e) :\equiv \mathrm{id}_{x \downarrow_e^{\mathscr{C}(x, \_)}}$$

By induction on e, $\mathrm{idd}(e)$ is a $\mathscr{C}$-equivalence with two-sided inverse $\mathrm{idd}(e^{-1})$.

Equivalence and univalence

So with the proof that $\mathrm{idd}(e)$ is a $\mathscr{C}$-equivalence, we get

$$x = y \xrightarrow{\mathrm{idtoeqv}_{\mathscr{C}}} x \simeq_{\mathscr{C}} y$$

in any wild category $\mathscr{C}$.

$\mathrm{idtoeqv}_{\mathscr{U}}$ is equal to the HoTT book definition of idtoeqv.

Equivalence and univalence

Definition

$\mathscr{C}$ is *univalent* if $\mathrm{idtoeqv}_{\mathscr{C}}$ is a (HoTT) equivalence.

Examples

- Univalent 1-categories $\mathscr{C}$
  ($\mathscr{C}$-equivalence and isomorphism are equivalent,
  $\mathscr{C}$-univalence is 1-categorical univalence)
- The universe $\mathscr{U}$
  ($\mathscr{U}$-equivalence is type theoretic equivalence,
  $\mathscr{U}$-univalence is type theoretic univalence)

Equivalence and univalence

In a univalent wild category $\mathscr{C}$, we have

$$\mathsf{ua}_{\mathscr{C}} : x \simeq_{\mathscr{C}} y \to x = y$$

satisfying

$$\mathsf{idtoeqv}_{\mathscr{C}}(\mathsf{ua}_{\mathscr{C}}(f, u)) = (f, u).$$

Abusing notation, we say

$$\mathsf{idd}(\mathsf{ua}_{\mathscr{C}}(f)) = f$$

whenever f is a $\mathscr{C}$-equivalence.

2-coherence

Familiar from bicategories:

Definition

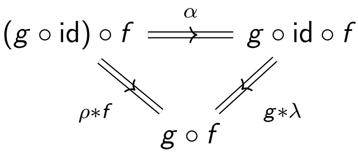
A wild category $\mathscr{C}$ has *triangle coherences* if for all

$$x \xrightarrow{f} y \xrightarrow{g} z$$

there is an equality of equalities

$$\triangle_{f,g} : \alpha \cdot (g * \lambda) = f * \rho$$

making the triangle commute:



2-coherence

Definition

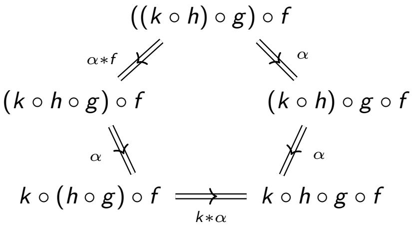
A wild category $\mathscr{C}$ has *pentagon coherences* if for all

$$v \xrightarrow{f} w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{k} z$$

there is an equality

$$\triangle_{f,g,h,k} : \alpha \cdot (g * \lambda) = f * \rho$$

filling the pentagon

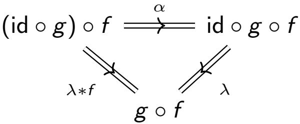


2-coherence

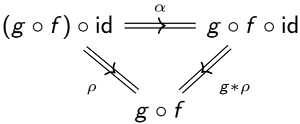
Definition

A wild category with triangle and pentagon coherences is a *2-coherent wild category*.

Usual coherences hold, e.g. the other triangle equalities



and



(Hart-Hou '23) call 2-coherent wild categories "bicategories".

Commuting squares

Generalize the theory of type theoretic pullbacks (Avigad, Kapulkin, Lumsdaine '15) to the 2-coherent wild setting.

Let $\mathscr{C}$ be a 2-coherent wild category.
A *cospan* c in $\mathscr{C}$ is an element

$$c \equiv (A, B, C, f, g) : \sum_{(A, B, C : \mathscr{C}_0)} \mathscr{C}(A, C) \times \mathscr{C}(B, C)$$

Diagrammatically,

$$c \equiv A \xrightarrow{f} C \xleftarrow{g} B$$

Commuting squares

Definition

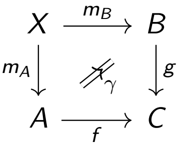
A *commuting square on c with source X* is an element

$$(m_A, m_B, \gamma)$$

of type

$$\mathrm{CommSq}_c(X) :\equiv \sum_{(m_A \,:\, \mathscr{C}(X,\, A))} \sum_{(m_B \,:\, \mathscr{C}(X,\, B))} f \circ m_A = g \circ m_B.$$

Diagrammatically,



The equality 2-cell γ is relevant in wild commuting squares.

Equality of commuting squares, 1

Characterizing the equality type of $\mathrm{CommSq}_c(X)$, we get:

Lemma

The equality

$$\begin{array}{ccc} X & \xrightarrow{m_B} & B \\ {\scriptstyle m_A}\downarrow & \overset{\gamma}{\nearrow\!\!\!\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{m_B{}'} & B \\ {\scriptstyle m_A{}'}\downarrow & \overset{\gamma'}{\nearrow\!\!\!\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

in $\mathrm{CommSq}_c(X)$ *is equivalent to the type of triples of equalities*

$$e_A : m_A = m_A{}'$$
$$e_B : m_B = m_B{}'$$
$$\eta : \gamma = (f * e_A) \cdot \gamma' \cdot (g * e_B)^{-1}$$

by e.g. the fundamental theorem of identity types.

Operations on commuting squares

Relevant to talking about split comprehension in wild cwfs later.

Definition (Pasting)

If $\quad \mathfrak{S} \;:\equiv\; \begin{array}{ccc} X & \xrightarrow{m_B} & B \\ {\scriptstyle m_A}\downarrow & \not\nearrow_{\gamma} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \quad$ and $\quad \mathfrak{S}' \;:\equiv\; \begin{array}{ccc} B & \xrightarrow{m_D} & D \\ {\scriptstyle g}\downarrow & \not\nearrow_{\gamma'} & \downarrow{\scriptstyle g'} \\ C & \xrightarrow[f']{} & E \end{array}$

are commuting squares, then

$$\mathfrak{S}|\mathfrak{S}' \;:\equiv\; \begin{array}{ccc} X & \xrightarrow{m_D \circ m_B} & D \\ {\scriptstyle m_A}\downarrow & \not\nearrow_{\gamma|\gamma'} & \downarrow{\scriptstyle g'} \\ A & \xrightarrow[f' \circ f]{} & E \end{array}$$

is a commuting square on cospan $(f' \circ f, g')$ and source X.

$$\gamma|\gamma' \;:\equiv\; \alpha \cdot (f' * \gamma) \cdot \alpha^{-1} \cdot (m_E * \gamma') \cdot \alpha$$

(not necessary to understand this talk)

Operations on commuting squares

Definition (Precomposition)

If

$$\mathfrak{S} \;:\equiv\; \begin{array}{ccc} X & \xrightarrow{m_B} & B \\ {\scriptstyle m_A}\downarrow & \overset{\gamma}{\nearrow\!\!\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

is a commuting square with source X and $m : \mathscr{C}(Y, X)$, there is a square

$$\mathfrak{S} \,\square\, m \;:\equiv\; \begin{array}{ccc} Y & \xrightarrow{m_B \circ m} & B \\ {\scriptstyle m_A \circ m}\downarrow & \overset{e}{\nearrow\!\!\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

where $e :\equiv \alpha \cdot (\gamma * m) \cdot \alpha^{-1}$.

Operations on commuting squares

Lemma

Precomposition gives a right monoid action of morphisms on commuting squares:

1. $\mathfrak{S} \square id = \mathfrak{S}$
2. $\mathfrak{S} \square (g \circ f) = \mathfrak{S} \square g \square f$.

Proof.

By unfolding definitions and calculating using:

- equations for whiskering
- the right identity triangle coherence for (1)
- the pentagon coherence for (2)

□

Operations on commuting squares

Suppose

$$\mathfrak{S} \;:\equiv\; \begin{array}{ccc} X & \xrightarrow{m_B} & B \\ {\scriptstyle m_A}\downarrow & \not\Rightarrow_{\gamma} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

and $e : X = Y$.
We can transport $\mathfrak{S}$ in the family

$$\mathsf{CommSq}_{(f,g)}(\_) : \mathscr{C}_0 \to \mathsf{Type}$$

along e to get a commuting square on the same cospan, but with source Y.

Operations on commuting squares

Corollary

$$\mathfrak{S}\downarrow_e^{\mathsf{CommSq}_{(f,g)}(\_)} = \mathfrak{S} \,\square\, \mathsf{idd}(e^{-1}).$$

Proof.

By induction on e, reduces to showing $\mathfrak{S} = \mathfrak{S} \,\square\, \mathsf{id}$. □

Equality of commuting squares, 2

Recall that if

$$c \;:\equiv\; A \xrightarrow{f} C \xleftarrow{g} B$$

and $X : \mathscr{C}_0$ then

$$\mathrm{CommSq}_c(X) \equiv \sum_{(m_A : \mathscr{C}(X,\, A))\,(m_B : \mathscr{C}(X,\, B))},\; f \circ m_A = g \circ m_B.$$

Take the total space over $X : \mathscr{C}_0$ and define

$$\mathrm{CommSq}(c) :\equiv \sum_{(X : \mathscr{C}_0)},\; \mathrm{CommSq}_c(X).$$

Equality of commuting squares, 2

Corollary

The equality

$$(X, \mathfrak{S}) = (X', \mathfrak{S}')$$

of elements of $\mathsf{CommSq}(c)$ *is equivalent to the type of pairs*

$$e : X = X'$$

and

$$H : \mathfrak{S} = \mathfrak{S}' \mathbin{\square} \mathsf{idd}(e).$$

Proof.

By the equality of Σ-types and the previous characterization of transport of commuting squares. □

Pullbacks

Definition

A commuting square

$$(P, \mathfrak{P}) \;:\equiv\; \begin{array}{ccc} P & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & \Nearrow_{\mathfrak{p}} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} : \mathrm{CommSq}(c)$$

is a *pullback* if the family of precomposition maps

$$\mathfrak{P} \,\square_{\_}\, \_ : \prod_{(X \,:\, \mathscr{C}_0)} \mathscr{C}(X, P) \to \mathrm{CommSq}_c(X)$$

is a family of equivalences.

So being a pullback is a proposition. Have

$$\text{is-pullback} : \mathrm{CommSq}(c) \to \mathrm{Type}.$$

Pullbacks

Examples

- Wild pullbacks in strict categories are strict 1-categorical pullbacks.
- Wild pullbacks in the universe wild categories $\mathcal{U}$ are type theoretic pullbacks (AKL '15).

Equality of pullbacks

Define the type of pullbacks on a cospan c,

$$\mathsf{Pullback}(c) :\equiv \sum_{(\mathfrak{P} : \mathsf{CommSq}(c))} \mathsf{is\text{-}pullback}(\mathfrak{P}).$$

Equality of pullbacks is thus equality of the underlying commuting squares.

Lemma

In a strict category, $\mathsf{Pullback}(c)$ *is a set for any cospan* c*.*

Proof.

Since $\mathsf{CommSq}(c)$ is a Σ-type of sets. □

Equality of pullbacks

Lemma

In a 2-coherent univalent wild category $\mathscr{C}$, $\mathsf{Pullback}(c)$ *is a proposition for any cospan* c.

Proof.

Suppose

$$(P, \mathfrak{P}), (P', \mathfrak{P}') : \mathsf{Pullback}(c).$$

Need an equality

$$h : P = P' \quad \text{such that} \quad \mathfrak{P} = \mathfrak{P}' \,\square\, \mathsf{idd}(h).$$

From centers of contraction of $(\mathfrak{P} \,\square\, \_)^{-1}(\mathfrak{P}')$ and $(\mathfrak{P}' \,\square\, \_)^{-1}(\mathfrak{P})$, get

$$m : \mathscr{C}(P,\, P'), \qquad m' : \mathscr{C}(P',\, P)$$

such that

$$\mathfrak{P}' \,\square\, m = \mathfrak{P} \quad \text{and} \quad \mathfrak{P} \,\square\, m' = \mathfrak{P}'.$$

Equality of pullbacks

Lemma

In a 2-coherent univalent wild category $\mathscr{C}$, Pullback(c) *is a proposition for any cospan c.*

Proof (cont.)

This implies m is a $\mathscr{C}$-equivalence with two-sided inverse m'. With univalence, take

$$h :\equiv \mathsf{ua}_{\mathscr{C}}(m) : P = P'.$$

Then

$$\mathfrak{P} = \mathfrak{P}' \,\square\, m = \mathfrak{P}' \,\square\, \mathsf{idd}(\mathsf{ua}_{\mathscr{C}}(m)) = \mathfrak{P}' \,\square\, \mathsf{idd}(h)$$

as required. □

Pullback pasting

The familiar pullback pasting lemma. In 2-coherent wild categories, need to construct a 3-cell between 2-cells.

Theorem

Suppose we have a diagram of commuting squares in a 2-coherent wild category

$$\begin{array}{ccc} A' & \xrightarrow{\ i\ } & A \\ {\scriptstyle f'}\downarrow & \Nearrow_{\mathfrak{q}} & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{\ j\ } & B \\ {\scriptstyle g'}\downarrow & \Nearrow_{\mathfrak{p}} & \downarrow{\scriptstyle g} \\ C' & \xrightarrow[\ k\ ]{} & C \end{array}$$

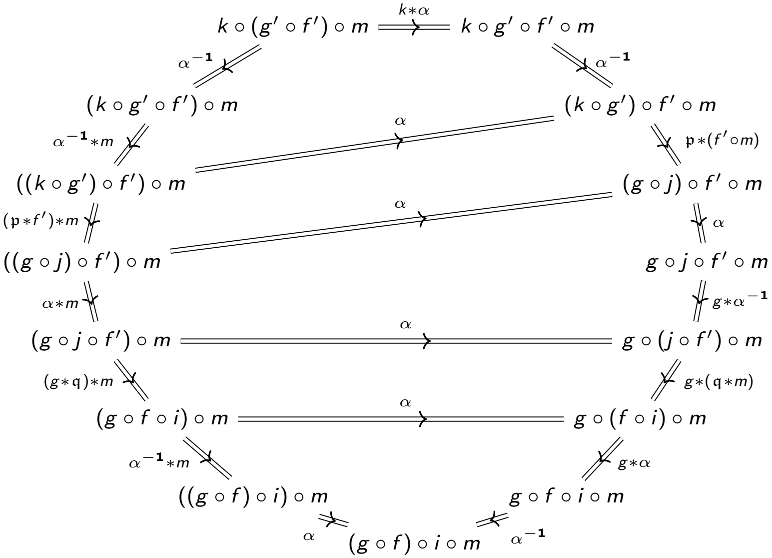
where the bottom square is a pullback. Then the top square is a pullback if and only if the pasting is a pullback.

Pullback pasting

Proof.

Uses a number of intermediate lemmas not shown (similar to (AKL '15)).

Ultimately comes down to showing the equality of certain 2-cells as in the outer boundary of the following coherence diagram:


k ∘ (g′ ∘ f′) ∘ m
k∗α
k ∘ g′ ∘ f′ ∘ m
α⁻¹
α⁻¹
(k ∘ g′ ∘ f′) ∘ m
(k ∘ g′) ∘ f′ ∘ m
α⁻¹∗m
α
𝔭∗(f′∘m)
((k ∘ g′) ∘ f′) ∘ m
(g ∘ j) ∘ f′ ∘ m
(𝔭∗f′)∗m
α
α
((g ∘ j) ∘ f′) ∘ m
g ∘ j ∘ f′ ∘ m
α∗m
g∗α⁻¹
(g ∘ j ∘ f′) ∘ m
α
g ∘ (j ∘ f′) ∘ m
(g∗𝔮)∗m
g∗(𝔮∗m)
(g ∘ f ∘ i) ∘ m
α
g ∘ (f ∘ i) ∘ m
α⁻¹∗m
g∗α
((g ∘ f) ∘ i) ∘ m
g ∘ f ∘ i ∘ m
α
(g ∘ f) ∘ i ∘ m
α⁻¹

Pullback pasting

Proof (cont.)

Insert associator 2-cells α judiciously to decompose into commuting regions:

- three pentagon coherences
- two commuting squares by properties of whiskering.

□

Get the horizontal pasting theorem by transposing pullback squares.

Part 2

Wild internal models of type theory

Develop Dybjer's categories with families (cwf's), generalized to wild categories of contexts. I assume familiarity with 1-categorical cwf's.

Brief recap, GAT of a 1-cwf:

Contexts	*Substitutions*		
$\Gamma, \Delta : \mathscr{C}_0$	$\sigma, \tau : \mathscr{C}(\Gamma, \Delta)$		
$\mathscr{C}$-types	*$\mathscr{C}$-terms*	*Substitution in types*	*Substitution in terms*
$\mathsf{Ty}\,\Delta : \mathsf{Type}$	$\mathsf{Tm}_{\Delta}\,A : \mathsf{Type}$	$A[\sigma]_{\mathsf{T}} : \mathsf{Ty}\,\Gamma$	$a[\sigma]_{\mathsf{t}} : \mathsf{Tm}_{\Gamma}\,(A[\sigma]_{\mathsf{T}})$
Context ext.	*Substitution ext.*	*Display map*	*Generic element*
$\Delta.A : \mathscr{C}_0$	$(\sigma\,, t) : \mathscr{C}(\Gamma, \Delta.A)$	$\mathsf{p}_A : \mathscr{C}(\Delta.A, \Delta)$	$\mathsf{q}_A : \mathsf{Tm}_{\Delta.A}\,(A[\mathsf{p}]_{\mathsf{T}\,A})$

appropriately quantified.

Higher cwfs in intensional type theory

GAT of a 1-cwf, cont.
Equations need transports along paths:

Functoriality

$[\mathrm{id}]_{\mathsf{T}} : A[\mathrm{id}]_{\mathsf{T}} = A$ $\qquad [\circ]_{\mathsf{T}} : A[\sigma \circ \tau]_{\mathsf{T}} = A[\sigma]_{\mathsf{T}}[\tau]_{\mathsf{T}}$

$[\mathrm{id}]_{\mathsf{t}} : a[\mathrm{id}]_{\mathsf{t}} = a_{\downarrow [\mathrm{id}]_{\mathsf{T}}^{-1}}$ $\qquad [\circ]_{\mathsf{t}} : a[\sigma \circ \tau]_{\mathsf{t}} = a[\sigma]_{\mathsf{t}}[\tau]_{\mathsf{t}\,\downarrow [\circ]_{\mathsf{T}}^{-1}}$

Context comprehension

$\mathrm{p}\beta : \mathrm{p}_A \circ (\sigma , t) = \sigma$ $\qquad \mathrm{q}\beta : q[\sigma , t]_{\mathsf{t}} = t_{\downarrow [=\mathrm{p}\beta]_{\mathsf{T}}^{-1} \cdot [\circ]_{\mathsf{T}}}$

$,\eta : (\mathrm{p}_A , \mathrm{q}_A) = \mathrm{id}_{\Gamma . A}$ $\qquad ,\circ : (\tau , t) \circ \sigma = (\tau \circ \sigma , t[\sigma]_{\mathsf{t}\,\downarrow [\circ]_{\mathsf{T}}^{-1}})$

For us, explicitly manipulating these transports is completely unavoidable.

Higher cwfs in intensional type theory

For us, explicitly manipulating these transports is completely unavoidable:

- Working in a higher setting: can no longer handwave transports away by appealing to conservativity of extensional TT over set-based fragments of HoTT.
- Want to find candidate coherence conditions: must look at the paths we're transporting along.

But in this talk I will mostly try to avoid burdening you with this.

Wild cwfs

Definition

A *wild cwf* $\mathscr{C}$ is a model of the GAT of a 1-cwf, where we replce the 1-category of contexts with a wild category of contexts.

Example

Any strict cwf is a wild cwf. In particular, the syntax QIIT (Altenkirch-Kaposi '16) is a wild cwf.

Wild cwfs

Example (Universe model)

Any universe wild categorical $\mathcal{U}$ can be given the structure of a wild cwf:

$$\begin{aligned} \mathsf{Ty}\,\Gamma &:\equiv \Gamma \to \mathcal{U} && \Gamma\text{-indexed type families} \\ \mathsf{Tm}_\Gamma\,A &:\equiv \Pi\,\Gamma\,A && \text{Sections of } A : \Gamma \to \mathcal{U} \end{aligned}$$

Substitution is function composition. If $\sigma : \Gamma \to \Delta$, then

$$\begin{aligned} A : \mathsf{Ty}\,\Delta \equiv \Delta \to \mathcal{U} &\quad\Longrightarrow\quad A[\sigma]_\mathsf{T} :\equiv A \circ \sigma : \Gamma \to \mathcal{U} \\ a : \mathsf{Tm}_\Delta\,A \equiv \Pi\,\Delta\,A &\quad\Longrightarrow\quad a[\sigma]_\mathsf{t} :\equiv a \circ \sigma : \Pi\,\Gamma\,(A \circ \sigma) \end{aligned}$$

Context extension is Σ, and
the equational components all hold by families of reflexivity proofs.

Structural properties of wild cwfs

Important structural properties of 1-cwfs already hold homotopically in wild cwfs.

Lemma

In any wild cwf $\mathscr{C}$, terms are sections of display maps.

$$\mathsf{Tm}_{\Gamma}\, A \simeq \sum_{(a\,:\,\mathscr{C}(\Gamma,\,\Gamma.A))} \mathsf{p} \circ a = id_{\Gamma}$$

Lemma

Substitutions into extended contexts are pairs.

$$\mathscr{C}(\Gamma,\, \Delta.A) \simeq \sum_{(\sigma\,:\,\mathscr{C}(\Gamma,\,\Delta))} \mathsf{Tm}\,(A[\sigma]_{\mathsf{T}})$$

Structural properties of wild cwfs

Corollary

If $\sigma : \mathscr{C}(\Gamma, \Delta.A)$ *is a substitution into an extended context then*

$$\sigma = (\mathsf{p} \circ \sigma\,, \mathsf{q}[\sigma]_{\mathsf{t}} \downarrow {[\circ]_{\mathsf{T}}}^{-1}).$$

That is, every substitution σ into an extended context is determined by its "initial segment"

$$\mathsf{p} \circ \sigma$$

and "last element"

$$\mathsf{q}[\sigma]_{\mathsf{t}}.$$

What we're aiming for

Classically, cwf's equivalent to full split comprehension categories.

We aim for a similar statement for sufficiently coherent wild cwfs, including the syntax and the universe models.

Concretely, we will prove the following theorem.

What we're aiming for

Theorem

Given a cospan $\Gamma \xrightarrow{\sigma} \Delta \xleftarrow{\mathsf{p}_A} \Delta . A$ in the category of contexts of a "2-precoherent" wild cwf $\mathscr{C}$, there is a weak pullback

$$\mathfrak{P}_{\sigma,A} \quad :\equiv \quad \begin{array}{ccc} \Gamma . A[\sigma]_{\mathsf{T}} & \xrightarrow{\sigma \,.\, A} & \Delta . A \\ {\scriptstyle \mathsf{p}}\downarrow & \mathsf{p}\beta^{-1} & \downarrow{\scriptstyle \mathsf{p}} \\ \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array}$$

When $\mathscr{C}$ is set-based or a universe, these squares are pullbacks for all σ and A, and further this choice of pullbacks is split...

What we're aiming for

Theorem

. . . for any pullbacks

$$\mathfrak{P}_{\sigma\circ\tau,A} \quad \equiv \quad \begin{array}{ccc} \mathsf{B}.\,A[\sigma\circ\tau]_{\mathsf{T}} & \xrightarrow{(\sigma\circ\tau)\,\cdot\,A} & \Delta.\,A \\ \downarrow\!\!\!\downarrow & \mathsf{p}\beta^{-1} & \downarrow\!\!\!\downarrow \\ \mathsf{B} & \xrightarrow[\sigma\circ\tau]{} & \Delta \end{array}$$

and

$$\mathfrak{P}_{\tau,A[\sigma]_{\mathsf{T}}} \mid \mathfrak{P}_{\sigma,A} \quad \equiv \quad \begin{array}{ccccc} \mathsf{B}.\,A[\sigma]_{\mathsf{T}}[\tau]_{\mathsf{T}} & \xrightarrow{\tau\,\cdot\,A[\sigma]_{\mathsf{T}}} & \Gamma.\,A[\sigma]_{\mathsf{T}} & \xrightarrow{\sigma\,\cdot\,A} & \Delta.\,A \\ \downarrow\!\!\!\downarrow & \mathsf{p}\beta^{-1} & \downarrow\!\!\!\downarrow & \mathsf{p}\beta^{-1} & \downarrow\!\!\!\downarrow \\ \mathsf{B} & \xrightarrow[\tau]{} & \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array}$$

the equality of pullbacks

$$\mathfrak{P}_{\sigma\circ\tau,A} = (\mathfrak{P}_{\tau,A[\sigma]_{\mathsf{T}}} \mid \mathfrak{P}_{\sigma,A})$$

is contractible.

What we're aiming for

Abstracting these properties will give us a definition of *2-coherent wild cwf* that we hope will be of further use.

Coherence for context extension

A technical excursion.

Lemma

Suppose

$$\Gamma \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \Delta . A$$

are parallel arrows into an extended context.
There is an equivalence $sub^{=}$ *that takes*

$$e_1 : \mathsf{p} \circ \sigma = \mathsf{p} \circ \tau$$

and

$$e_2 : \mathsf{q}[\sigma]_{\mathsf{t}\downarrow_{[\circ]_{\mathsf{T}}{}^{-1}\cdot[{=}e_1]_{\mathsf{T}}\cdot[\circ]_{\mathsf{T}}}} = \mathsf{q}[\tau]_{\mathsf{t}},$$

and gives an equality

$$sub^{=}(e_1, e_2) : \sigma = \tau.$$

Intuitively, this is because of the previous corollary: extended substitutions are determined by their initial segment and last element.

Coherence for context extension

Note the following composition of maps:

$$
\begin{aligned}
& \sum (e : \mathsf{p} \circ \sigma = \mathsf{p} \circ \tau),\ \mathsf{q}[\sigma]_{\mathsf{t}} \downarrow_{[\circ]_{\mathsf{T}}{}^{-1} \cdot [=e]_{\mathsf{T}} \cdot [\circ]_{\mathsf{T}}} = \mathsf{q}[\tau]_{\mathsf{t}} \\
\xrightarrow{\mathsf{sub}^{=}} \quad & \sigma = \tau \\
\xrightarrow{\mathsf{p}*\_} \quad & \mathsf{p} \circ \sigma = \mathsf{p} \circ \tau
\end{aligned}
$$

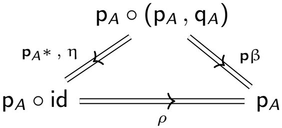
It will be useful later if this composition were equal to the first projection.
Motivates the following definition.

Coherence for context extension

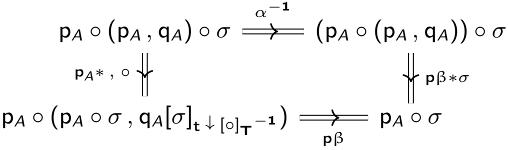
Recall the cwf equations

$$\mathsf{p}\beta : \mathsf{p} \circ (\sigma \,, t) = \sigma$$
$$, \eta : (\mathsf{p} \,, \mathsf{q}) = \mathsf{id}$$
$$, \circ : (\tau \,, t) \circ \sigma = (\tau \circ \sigma \,, t[\sigma]_{\mathsf{t} \downarrow [\circ]_{\mathsf{T}}{}^{-1}})$$

They interact with unitors and associators of the underlying wild category of contexts in the following diagrams:



and



Coherence for context extension

Definition

We say that a wild cwf $\mathscr{C}$ has *coherators for context extension* if every diagram of the previous two forms commutes.

Examples

Cwfs with precategories of contexts automatically satisfy coherences on 2-cells. In universe cwfs, the diagrams above reduce to composites of *refl* paths, and commute definitionally.

Coherence for context extension

Lemma

If $\mathscr{C}$ has coherators for context extension then the composition

$$\sum (e : \mathsf{p} \circ \sigma = \mathsf{p} \circ \tau),\ \mathsf{q}[\sigma]_{\mathsf{t}\downarrow [\circ]_{\mathsf{T}}{}^{-1} \cdot [{}^{=}e]_{\mathsf{T}} \cdot [\circ]_{\mathsf{T}}} = \mathsf{q}[\tau]_{\mathsf{t}}$$

$$\xrightarrow{sub^{=}} \quad \sigma = \tau$$

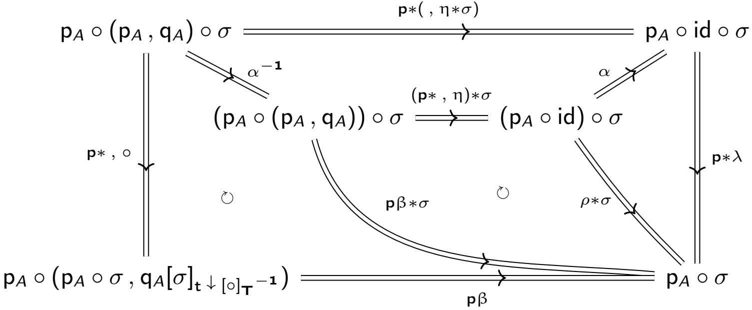
$$\xrightarrow{\mathsf{p}*_{=}} \quad \mathsf{p} \circ \sigma = \mathsf{p} \circ \tau$$

is equal to the first projection.

Coherence for context extension

Proof.

Technical and calculation-heavy, involves this coherence pasting diagram:



□

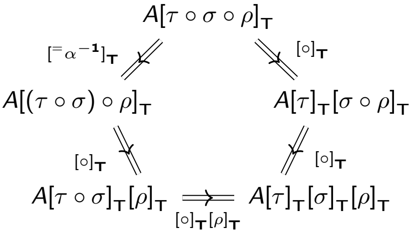
Type pentagonators

Type substitution also interacts with the associators of the category of contexts:

For all contexts and substitutions

$$\Gamma \xrightarrow{\rho} \Delta \xrightarrow{\sigma} \mathrm{E} \xrightarrow{\tau} \mathrm{Z}$$

and $\mathscr{C}$-types $A : \mathrm{Ty}\,\mathrm{Z}$, there is the pentagon of canonical equalities



Type pentagonators

Definition

A wild cwf $\mathscr{C}$ has *type pentagonators* if all equality pentagons of the previous form commute.

Examples

Set-based and universe cwfs have trivial type pentagonators (in different ways).

2-precoherent cwfs

Now we can define

Definition

A wild cwf $\mathscr{C}$ is *2-precoherent* if it has

- a 2-coherent category of contexts (i.e. triangle and pentagon coherences for ρ, λ, α),
- type pentagonators,
- coherators for context extension.

Wild substitution in types

Theorem

Suppose that $\mathscr{C}$ is a 2-precoherent cwf, $\sigma : \mathscr{C}(\Gamma, \Delta)$ a substitution and $A : \mathsf{Ty}\,\Delta$. Then the substitution

$$\sigma^{\cdot A} :\equiv (\sigma \circ \mathsf{p}_{A[\sigma]_\mathsf{T}} , \mathsf{q}_{A[\sigma]_\mathsf{T}} \downarrow^{\mathsf{Tm}}_{[\circ]_\mathsf{T}}{}^{-1})$$

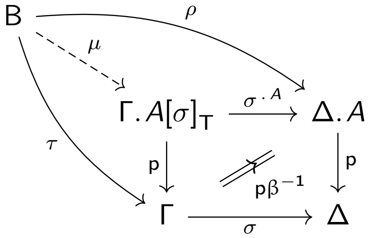
makes the commuting square

$$\mathfrak{P}_{\sigma,A} \quad :\equiv \quad \begin{array}{ccc} \Gamma.\, A[\sigma]_\mathsf{T} & \xrightarrow{\sigma^{\cdot A}} & \Delta.\, A \\ \mathsf{p}\big\downarrow & \nearrow\!\!\!\!\!\!\nearrow\ \mathsf{p}\beta^{-1} & \big\downarrow\mathsf{p} \\ \Gamma & \xrightarrow[\sigma]{} & \Delta \end{array}$$

a weak pullback in $\mathscr{C}$.

Wild substitution in types

This means that for every outer commuting square $\mathfrak{S}$ as in



with $\gamma : \sigma \circ \tau = \mathsf{p}_A \circ \rho$, there is a mediating substitution

$$\mu : \mathscr{C}(\mathsf{B}, \Gamma. A[\sigma]_\mathsf{T})$$

such that

$$\mathfrak{P}_{\sigma,A} \square \mu = \mathfrak{S}.$$

Wild substitution in types

Proof sketch.

In the following, all smiley faces are equalities.

1. Follow your nose and define

$$\mu :\equiv (\tau\,, \mathsf{q}[\rho]_{\mathsf{t}} \downarrow ☺)$$

2. By characterization of equality of pullbacks, to show that

$$\mathfrak{P}_{\sigma,A} \,\square\, \mu = \mathfrak{S}$$

we can construct equalities

$$\delta : \mathsf{p} \circ \mu = \tau$$

$$\epsilon : \sigma^{\cdot A} \circ \mu = \rho$$

such that there is an equality of 2-cells

$$\alpha^{-1} \cdot (\mathsf{p}\beta^{-1} * \mu) \cdot \alpha = (\sigma * \delta) \cdot \gamma \cdot (\mathsf{p} * \epsilon)^{-1}.$$

Wild substitution in types

Proof sketch (cont.)

3. By definition of μ, can take

$$\delta :\equiv \mathsf{p}\beta : \mathsf{p} \circ \mu = \tau.$$

4. $\epsilon : \sigma \cdot^{A} \circ \mu = \rho$ needs to be an equality of extended substitutions. Define

$$\epsilon :\equiv \mathsf{sub}^{=}(\epsilon_0, \epsilon_1),$$

for which we need equalities

$$\epsilon_0 : \mathsf{p} \circ \sigma \cdot^{A} \circ \mu = \mathsf{p} \circ \rho$$

and

$$\epsilon_1 : \mathsf{q}[\sigma \cdot^{A} \circ \mu]_{\mathsf{t}} \downarrow \text{😐}_{(\epsilon_0)} = \mathsf{q}[\rho]_{\mathsf{t}}.$$

Wild substitution in types

Proof sketch (cont.)

5. We can construct

$$\epsilon_0 :\equiv \text{☺}$$

   By calculation, constructing ϵ_1 is equivalent to showing that

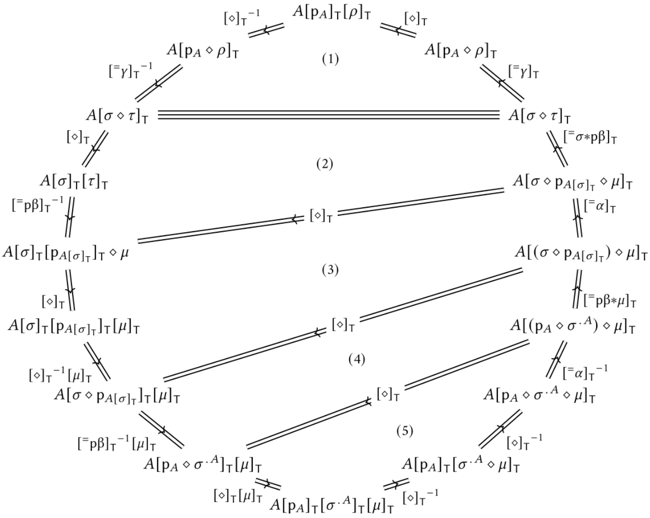
$$\mathsf{q}[\rho]_{\mathsf{t}\downarrow \text{☹}(\text{☺})} = \mathsf{q}[\rho]_{\mathsf{t}}.$$

6. We do this by showing that

$$\text{☹} = \mathsf{refl}.$$

   Some path algebra shows that ☹ is equal to the outer boundary of the following diagram:

Wild substitution in types



. . . which decomposes into type pentagonators and commuting squares.

Wild substitution in types

Proof sketch (cont.)

7. That completes the construction of δ and ϵ. What's left is to show that

$$\alpha^{-1} \cdot (\mathsf{p}\beta^{-1} * \mu) \cdot \alpha = (\sigma * \mathsf{p}\beta) \cdot \gamma \cdot (\mathsf{p} * \mathsf{sub}^{=}(\epsilon_0, \epsilon_1))^{-1}.$$

Simplifying

$$\mathsf{p} * \mathsf{sub}^{=}(\epsilon_0, \epsilon_1) = \epsilon_0$$

on the right hand side, the rest becomes a straightforward calculation.

□

Wild substitution in types

So substitution in types is weak pullback in 2-precoherent cwfs.

Lemma

If $\mathscr{C}$ is set-based, the $\mathfrak{P}_{\sigma,A}$ are pullbacks.

Proof.

Using the fact that $\mathrm{CommSq}_c(X)$ are sets in precategories. □

Lemma

If $\mathscr{C} = \mathfrak{U}$ is a universe wild cwf, the $\mathfrak{P}_{\sigma,A}$ are pullbacks.

Proof.

By instantiating the type theoretic pullback and Corollary 4.1.9 of (AKL '15). □

Split comprehension

Theorem

Suppose $\mathscr{C}$ is a set-based or universe wild cwf. Then for all

$$\mathsf{B} \xrightarrow{\tau} \Gamma \xrightarrow{\sigma} \Delta$$

and $A : \mathrm{Ty}\,\Delta$, the equality of pullbacks

$$\mathfrak{P}_{\sigma\circ\tau,A} = (\mathfrak{P}_{\tau,A[\sigma]_{\mathsf{T}}} \mid \mathfrak{P}_{\sigma,A})$$

is contractible.

Proof.

For any cospan c, Pullback(c) is always a set in set-based and 2-coherent univalent wild categories. So the equality is a proposition. It is inhabited by calculation and a simple lemma I'll elide in this talk. □

2-coherent wild cwfs

Abstracting these properties, we arrive at a candidate

Definition

A *2-coherent* wild cwf is a 2-precoherent cwf such that

- for every σ and A, $\mathfrak{P}_{\sigma,A}$ is a pullback
- for every τ, σ and A,

$$\mathfrak{P}_{\sigma\circ\tau,A} = (\mathfrak{P}_{\tau,A[\sigma]_{\mathsf{T}}} \mid \mathfrak{P}_{\sigma,A})$$

  is contractible.

Examples

Any set-based cwf, and any universe wild cwf.

Further investigations

- Are 2-coherent cwfs a good notion of "higher cwf with hom-groupoids"?
- Determine the initial model of 2-coherent cwfs
- Study wild *natural models* instead of wild cwfs
- For me, use 2-coherent wild cwfs to organize the construction of Reedy fibrant inverse diagrams.